COMPUTABLE CHOICE FUNCTIONS FOR COMPUTABLE LINEAR ORDERINGS

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1. Introduction

One direction of research in computability theory is the study of the effective content of combinatorial theorems. The last chapter of Rosenstein's book [2] reports on this type of investigation for linear orderings. Questions raised in that chapter have been the subject of a number of papers, and a summary of results in the area can be found in Downey [1]. In this paper, we consider the problem of finding an effective translation of the fact that every linear ordering has a choice set by characterizing the computable linear order-types which hereditarily have arbitrarily large blocks, for which all computable models have computable choice sets; in this case, we say that the order-type *admits computable choice sets.* The problem of finding choice sets for a linear ordering is equivalent to that of finding maximal dense subsets when all condensation classes of the linear ordering are finite. These problems also seem to be somewhat related to the problem of characterizing the computable linear orderings which admit non-trivial self-embeddings (one-to-one endomorphisms). The naturally conjectured condition for a computable linear order-type to admit computable choice sets is that it can be written as a finite separated sum $\alpha_0 + 1 + \alpha_1 + 1 + \cdots + 1 + \alpha_n$ where each component α_i is either of the form $n \cdot \eta$ or has finitely many condensation classes (η is the order-type of the rational numbers). We show that this condition is, in fact, necessary and sufficient, when every finite separation of the linear ordering has a component with infinitely many maximal blocks and arbitrarily large blocks.

The proof of our result is highly non-uniform. We present two constructions. The first is a construction, with parameters, of computable models which do not have computable choice sets, and through different choices for parameters, apply this construction to two different classes of order-types. The second construction handles the remaining case. We then show that any linear order-type in the class under consideration which does not satisfy the above condition has a closed interval in one of the classes, and so a model without a computable choice set. Each construction fixes an order-type α, assumes that there is a model of α which has a computable choice set, and builds a computable model isomorphic to the given one which fails to have a computable choice set; the choice set for the first model is a crucial ingredient for ensuring the success of that construction. The first construction is fairly standard and builds a Δ^0_2 isomorphism, but the second is more complex and builds a Δ^0_3 isomorphism.

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We will also mention some results we have obtained when there is a finite bound on the size of maximal blocks of α. As we have no necessary and sufficient condition in this case, we will not give proofs, some of which are modifications of our second construction.

We will use standard interval notation when talking about linear order-types or linear orderings.

Definition 1.1. Let α be a countable linear order-type. A *block* of α (also known as a *scattered subinterval* of α) is a subinterval B of α such that for all $a, b \in B$, $|[a, b]| < \infty$. B is a *maximal block* of α if B is a block of α and no proper superset of B is a block of α. (Maximal blocks are also called *condensation classes* or *maximal scattered subintervals*.) A block of cardinality n is called an *n-block*, a block of cardinality $\leq n$ is called a *$\leq n$-block*, etc. α is *n-bounded* if all of its maximal blocks are $\leq n$-blocks, is *$\langle k, n\rangle$-bounded* if it is n-bounded and all of its maximal blocks are $\geq k$-blocks, and is *unbounded* if it is not n-bounded for any n. α is *n-constant* if all of its maximal blocks are n-blocks. α is *condensationally finite* if it has only finitely many maximal blocks. Finally, α is *completely infinite* if it is not condensationally finite and does not have any finite maximal blocks. These definitions carry over from order-types to linear orderings in the obvious way. We call a linear ordering of order-type α a *model* of α. α^* is the order-type obtained by reversing the ordering of α.

Definition 1.2. Let α be a countable linear order-type. α is the *separated sum* of $\{\alpha_i : i \leq n\}$ if $\alpha = 1 + \alpha_0 + 1 + \alpha_1 + 1 + \cdots + 1 + \alpha_n + 1$, and in this case, we say that α_i and α_{i+1} are *separated*, call the points which are not in any α_i *separating points*, and call the α_i *components* of the separation. α is *finite-constant* if there is a finite separated sum of $\{\alpha_i\}$ of α such that for every i, α_i is either condensationally finite or n-constant for some n. Similarly, α is *finite-bounded* if there is a finite separated sum of $\{\alpha_i\}$ of α such that for every i, α_i is either condensationally finite or n-bounded for some n. These definitions carry over from order-types to linear orderings in the obvious way.

We will be dealing with properties of order-types which are inherited from an order-type α by some interval of every separated sum of α. We introduce terminology to describe such properties.

Definition 1.3. Let P be a property of order-types. We say that an order-type α is *hereditarily P* if given any finite separation $1 + \alpha_0 + 1 + \alpha_1 + 1 + \cdots + 1 + \alpha_n + 1$ of α, then there is an $i \leq n$ such that $1 + \alpha_i + 1$ has property P.

Definition 1.4. Let $\mathcal{L} = \langle L, \leq\rangle$ be a computable model of the order-type α. A *choice set* for $\mathcal{L}$ is a subset of L containing exactly one element from each maximal block. A *choice function* is the characteristic function of a choice set. α *admits computable choice sets* if each computable model $\mathcal{M}$ of α has a computable choice set.

In Section 2, we will show that if α is finite-constant, then α admits computable choice sets. The partial converse will employ two constructions. The requirements for these constructions will be presented in Section 3, and a general overview of the constructions will be presented in Section 4. Section 5 will present the construction in which the focus of the satisfaction of requirements is on large blocks, and Section 6 will present the construction in which the focus of the satisfaction of requirements is on small blocks. In each of these sections, once the construction is presented, we show how to apply it to several classes of order-types. The proof of our theorem also appears in Section 6.

Our notation is standard, for the most part. $|A|$ will denote the cardinality of the set A. A *string* is a finite sequence of symbols from an underlying set, and $\subseteq$ is the extension ordering for strings. $|\sigma|$ is the length of the string σ, i.e., the cardinality of its domain. If $|\sigma| > 0$ then we set $\sigma^- = \sigma \upharpoonright |\sigma| - 1$. $\sigma|\tau$ denotes that σ and τ are incomparable. Given strings σ and τ, the string $\sigma^\frown\tau$ will denote the *concatenation* of σ and τ, i.e., the string of length $|\sigma| + |\tau|$ which begins with σ and ends with τ.

2. Sufficiency

In this section, we show that every finite-constant computable order-type α admits computable choice sets. We will present lemmas covering some special cases first, and then prove the full result. Fix a computable order-type α.

Our first lemma shows that a computable choice set for a model of α is also a choice set for the corresponding model of α^*.

Lemma 2.1. *Let $\mathcal{M} = \langle M, \leq \rangle$ be a model of α, and let $\mathcal{M}^* = \langle M, \geq \rangle$ (i.e., the ordering of $\mathcal{M}^*$ is the reverse of the ordering of $\mathcal{M}$). Suppose that S is a choice set for $\mathcal{M}$. Then S is a choice set for $\mathcal{M}^*$.*

Proof. The blocks of $\mathcal{M}$ coincide with the blocks of $\mathcal{M}^*$. □

Our next lemma shows that if we express α as a finite separated sum, then α admits computable choice sets iff each of the summands admits computable choice sets.

Lemma 2.2. *Suppose that $\alpha = 1 + \alpha_0 + 1 + \alpha_1 + \cdots + 1 + \alpha_k + 1$. Then α admits computable choice sets iff, for all $i \leq k$, $1 + \alpha_i + 1$ admits computable choice sets.*

Proof. First suppose that α admits computable choice sets. For each $i \leq k$, let $\mathcal{M}_i = \langle M_i, \leq_{M_i} \rangle$ be a computable model of α_i. Then $1 + M_0 + 1 + \cdots + 1 + M_k + 1$, ordered by the composite ordering $\leq_M$, produces a computable model $\mathcal{M} = \langle M, \leq_M \rangle$ of α. By hypothesis, $\mathcal{M}$ has a computable choice set S. Then $S_i = S \upharpoonright M_i$ is computable, and can only fail to be a choice set for $\mathcal{M}_i$ if it fails to choose elements from blocks containing the endpoints of M_i. But we can redefine S_i to include any (or both) of the endpoints, if necessary, thereby obtaining a choice set for $\mathcal{M}_i$. As this modification differs only finitely from the original S_i, it is also computable.

Conversely, suppose that each α_i admits a computable choice set. Let $\mathcal{M} = \langle M, \leq_M \rangle$ be a computable model of α, where $M = 1 + M_0 + 1 + \cdots + 1 + M_k + 1$, and each M_i is the universe of a model of α_i under the induced ordering $\leq_{M_i}$. By hypothesis, we have choice sets S_i for each $\langle M_i, \leq_{M_i} \rangle$. Define S to be the union of these choice sets, together with the endpoints of each M_i. Then S is computable, and can only fail to be a choice set for $\mathcal{M}$ if it chooses two or three points in some of the blocks containing the separating points. As there are only finitely many separating points, there are only finitely many such blocks. We can finitely modify S to choose just one point from each of these blocks, thereby obtaining a computable choice set for $\mathcal{M}$. □

The following immediate corollary to the proof of Lemma 2.2 allows us to assume that α has both left and right endpoints.

Corollary 2.3. *α admits computable choice sets iff $1+\alpha+1$ admits computable choice sets.* □

Our next lemma handles the n-constant case.

Lemma 2.4. *Suppose that α is n-constant for some n. Then α admits computable choice sets.*

Proof. Let $\mathcal{M} = \langle M, \leq_M \rangle$ be a model of α. By Corollary 2.3, we may assume that $\alpha = n \cdot (1 + \eta + 1)$. Fix a computable enumeration $\langle M^s, \leq_M^s \rangle$ of $\mathcal{M}$ with least element 0 and greatest element 1. We will define another computable enumeration $\widehat{M}^s$ of M and a computable choice function g^s with domain $\widehat{M}^s$ which satisfies the following properties for $a, b \in \widehat{M}^s$:

$$a \leq_M^s b \;\&\; g^s(a) = g^s(b) = 1 \;\rightarrow\; |[a,b]| > n. \tag{2.1}$$

$$(a \leq_M^s b \;\&\; \forall c(c \in [a,b] \;\rightarrow\; g^s(c) = 0)) \;\rightarrow\; |[a,b]| < n. \tag{2.2}$$

$$s \in \widehat{M}^s. \tag{2.3}$$

Then for each maximal block B of M, it follows from (2.1) and (2.3) that $g(c) = 1$ for at most one $c \in B$, and from (2.2) and (2.3) that $g(c) = 1$ for at least one $c \in B$; thus g will be a choice function for $\mathcal{M}$.

The definitions of $\widehat{M}^s$ and g^s proceed by induction on s.

Stage 0: Let $\widehat{M}^0 = \{k : k \leq n\}$. Define $g^0(0) = g^0(1) = 1$ and $g^0(m) = 0$ if $2 \leq m \leq n$.

Stage $s > 0$: If $s \in \widehat{M}^{s-1}$, then $\widehat{M}^s = \widehat{M}^{s-1}$ and $g^s = g^{s-1}$. Suppose that $s \notin \widehat{M}^{s-1}$. Fix the greatest element $a \in M^{s-1}$ such that $a \leq_M s$ and $g^{s-1}(a) = 1$, and the smallest element $b \in M^{s-1}$ such that $s \leq_M b$ and $g^{s-1}(b) = 1$. Note that as $0 \leq_M s \leq_M 1$ and $g^{s-1}(0) = g^{s-1}(1) = 1$, a and b will exist. By (2.2) $|M^{s-1} \cap (a,b)| < n$. Speed up the enumeration of M to find $2n-1$ elements $a_1 \leq_M \cdots \leq_M a_{2n-1} \in (a,b)$ including all those in M^{s-1} such that $a_n \notin M^{s-1}$. The order-type α ensures that $|(a,b) \cap M| = \infty$, so such elements can be found. Define $\widehat{M}^s = \widehat{M}^{s-1} \cup \{a_1, \dots, a_{2n-1}\}$ and extend g^{s-1} to g^s by defining $g^s(a_i) = 0$ if $i \neq n$ and $g^s(a_n) = 1$. (2.1)–(2.3) are easily verified. □

Our sufficient condition is now easily proved.

Theorem 2.5. *Suppose that α is finite-constant. Then α admits computable choice sets.*

Proof. As α is finite-constant, so is $1+\alpha+1$, so we can express $1+\alpha+1$ as $1 + \alpha_0 + 1 + \alpha_1 + 1 \cdots + 1 + \alpha_k + 1$ where each α_i is either condensationally finite or n-constant for some n. Note that as the components α_i are separated, each is a computable order-type. If α_i is condensationally finite, then it has only finitely many maximal blocks, so admits computable choice sets; and if α_i is n-constant for some n, then it follows from Lemma 2.4 that α_i admits computable choice sets. The theorem now follows from Lemma 2.2 and Corollary 2.3. □

3. Necessity: Requirements and Strategies

We will present two constructions of computable models failing to have computable choice sets. The first covers the case in which we can implement a *large block strategy*, and the second covers the case in which can implement a *small block strategy*.

Fix a computable linear ordering $\mathcal{M} = \langle M, \leq_M \rangle$ of order-type α. Without loss of generality, we will assume that $\mathcal{M}$ has a computable choice function f. We will construct a computable linear ordering $\widetilde{\mathcal{M}} = \langle \widetilde{M}, \leq_{\widetilde{M}} \rangle$ and an isomorphism map $g : M \to \widetilde{M}$, and diagonalize against all possible computable choice functions. There will be three types of requirements; diagonalization requirements, requirements that g be total, and requirements that g be onto; our constructions will automatically ensure that g is one-to-one and order-preserving. We use the set N of natural numbers as the universe for both models so $M = \widetilde{M} = N$. However, we will need computable approximations to both models, and these may have different universes at a given stage. Thus we fix a computable sequence $\langle M^s : s \in N \rangle$ of finite sets whose union is M and an ordering $\leq^s_M$ such that at stage s, $\langle M^s, \leq^s_M \rangle$ is a sub-linear ordering of $\mathcal{M}$; and we will construct a computable sequence $\langle \widetilde{M}^s, \leq^s_{\widetilde{M}} \rangle$ of increasing finite linear orderings whose union is $\widetilde{\mathcal{M}}$. Let $\{\phi_n : n \in N\}$ be a computable listing of all computable partial functions. Our requirements are:

$$P_n : n \in \operatorname{dom}(g);$$
$$Q_n : n \in \operatorname{rng}(g);$$
$$R_n : \phi_n \text{ total } \to \text{ there is a maximal block } B \text{ of } \widetilde{\mathcal{M}} \text{ such that } |\{x \in B : \phi_n(x) = 1\}| \neq 1.$$

At each stage of a construction, we will have constructed a finite portion of the isomorphism map. The satisfaction of requirements P_n and Q_n will involve procedures which finitely extend the portion of the isomorphism map which has been specified before reaching the point at which we try to satisfy the requirement. There are two radically different strategies, however, to try to satisfy R_n when ϕ_n is total:

- Create a block $\widetilde{B}$ of $\widetilde{M}$ such that $\phi_n(\widetilde{a}) = \phi_n(\widetilde{b}) = 1$ for some $\widetilde{a} < \widetilde{b} \in \widetilde{B}$.
- Create a maximal block $\widetilde{B}$ of $\widetilde{M}$ such that $\phi_n(\widetilde{b}) \neq 1$ for all $\widetilde{b} \in \widetilde{B}$.

The first gives rise to the *large block construction*, and the second to a *small block construction*. The *large block strategy* requires that we wait until we find a suitable pair $\widetilde{a} \leq_{\widetilde{M}} \widetilde{b}$ such that $\phi_n(\widetilde{a}) = \phi_n(\widetilde{b}) = 1$. When found, the cardinality of the interval $[\widetilde{a}, \widetilde{b}]$ will be finite, but may be arbitrarily large. In order for this block to be the image of a block of M, the interval of M in which we try to find a preimage for $[\widetilde{a}, \widetilde{b}]$ must have arbitrarily large (possibly infinite) blocks. Thus this strategy can only be employed when α has large blocks. Furthermore, we will need a Δ^0_2-procedure which will allow us to identify a preimage which can be used to satisfy R_n without violating the isomorphism requirements.

The second strategy (the *small block strategy*) requires us to find a small maximal interval $\widetilde{B} \subset \widetilde{M}$ on which $\phi_n(\widetilde{c}) \neq 1$ for all $\widetilde{c} \in \widetilde{B}$. We begin by finding a small interval $B \in M$ and a larger interval $C \in M$, and specifying an image $\widetilde{C}$ for C. We now wait for $\phi_n(x)$ to converge for all $x \in \widetilde{C}$. If $|\{x \in \widetilde{C} : \phi_n(x) = 1\}| \neq 1$, then nothing more needs to be done. Otherwise, we revise $\widetilde{C}$ to $\widetilde{D}$ so that the unique $x \in \widetilde{C}$ such that $\phi_n(x) = 1$ is the first element of $\widetilde{D}$ and wait for $\phi_n(y)$ to converge for all $y \in \widetilde{D}$. If $|\{x \in \widetilde{D} : \phi_n(x) = 1\}| = 1$, then we have an interval $\widetilde{B} \subset \widetilde{D}$ of cardinality $|B|$ on which ϕ_n fails to take the value 1; we now revise the isomorphism map so that $\widetilde{B}$ is the image of B. The success of such a strategy relies on our ability to locate an interval on which the above strategy will succeed, and blocks within the interval to which we can apply the strategy without violating the isomorphism maps specified by higher priority requirements. The requirement that we preserve our ability to build an isomorphism will be satisfied by using the large blocks to store blocks which are specified by higher priority guesses at the isomorphism map. The interval which we seek will thus need to have infinitely many small blocks and infinitely many larger blocks; the construction will succeed when we have a Δ^0_3-procedure which identifies the intervals to which the strategy is to be applied. The definitions of *small* and *large* will depend on α.

4. Building Models without Choice Sets

We now introduce the general framework within which the first construction of this paper will take place. We will always start with a computable linear ordering $\mathcal{M} = \langle M, \leq_M \rangle$, and will assume that this linear ordering has a computable choice function; and we will construct a computable linear ordering $\widetilde{\mathcal{M}} = \langle \widetilde{M}, \leq_{\widetilde{M}} \rangle$ which is isomorphic to $\mathcal{M}$ and fails to have a computable choice function. At stage s of the construction, we will be given a finite subset M^s of M and will build a finite subset $\widetilde{M^s}$ which will be extended to $\widetilde{M}$ at the end of the construction. In this section, we analyze the situations for which such extensions are taken.

A linear ordering will be identified with its universe; we expect that this will not cause confusion. The typical situation starts with finite linearly ordered sets $A \subset M$ and $\widetilde{A}$, such that M and A share the same least and greatest elements and M has a computable choice function f. We will want to build a computable linearly ordered set $\widetilde{M} \supset \widetilde{A}$ which shares the least and greatest elements of $\widetilde{A}$ such that $M \cong \widetilde{M}$. In order to simplify the presentation, all linear orderings will be assumed to have least element 0 and greatest element 1, and extensions of linear orderings must preserve 0 and 1. Furthermore, for a given linear ordering $\langle A, \leq_A \rangle$, we will replace $\leq_A$ with $\leq$, thereby using the same symbol in different ways for different linearly ordered sets. The context will always be clear when this is done.

The finite approximations to the isomorphism map will be called *partial isomorphisms.*

Definition 4.1. Let A and B be linearly ordered sets such that B is finite. A *partial isomorphism* of A with B is a finite, one-to-one, order-preserving map g from A to B such that $g(0)\downarrow = 0$ and $g(1)\downarrow = 1$.

The sets $\mathrm{dom}(g)$ and $\mathrm{rng}(g)$ will induce partitions of $A - \mathrm{dom}(g)$ and $\widetilde{A} - \mathrm{rng}(g)$, respectively, into finitely many intervals on which we will have to act somewhat independently when extending either $\widetilde{M}$ or g. We introduce terminology for these partitions.

Definition 4.2. Let A be a linearly ordered set. A *separation* of A is a finite increasing sequence P of elements of A containing 0 and 1. The *length* of the separation is $|P|-1$, and measures the number of intervals determined by the separation. The pair $\langle P, \widetilde{P}\rangle$ is called a *separation* of $\langle A, \widetilde{A}\rangle$ if P and $\widetilde{P}$ are separations of A and $\widetilde{A}$ respectively and $|P| = |\widetilde{P}|$. Note that a finite partial isomorphism g of A with $\widetilde{A}$ induces a separation $\langle \mathrm{dom}(g), \mathrm{rng}(g)\rangle$ of $\langle A, \widetilde{A}\rangle$. We call this separation of $\langle A, \widetilde{A}\rangle$ the *separation induced by* g; g induces separations of each component in the same way.

Given a separation $P = \langle p_i : i \leq n\rangle$ of A where the ordering of the elements of P is that produced by the linear ordering of A, we let $I_i^A = [p_i, p_{i+1}] \cap A$. We will automatically use this notation without further comment; if A is adorned with subscripts, superscripts, etc, then the elements of P and the intervals I_i^A will be similarly adorned.

Our constructions will place restrictions on extensions of $\widetilde{A}$, requiring that certain intervals of $\widetilde{A}$ be preserved as blocks. We will also need to identify the intervals which can be certified as being infinite. Thus each partial isomorphism g will be accompanied by a *character* χ_g whose role is to describe the character of the interval. Characters will be maps into $\{0, 1, 2\}$, with 0 designating that no determination of the character of the interval has yet been made, 1 designating that the interval is to be preserved as a block, and 2 designating that we can certify that the interval is infinite. As a character χ_g is not uniquely definable from g, we will always accompany a partial isomorphism g with a character h having the same subscripts and superscripts. h will determine the character of the intervals produced by the separation induced by g. Before proceeding to the formal definition of characters, we first need to describe the process for certifying that an interval is infinite.

Definition 4.3. Fix $a, b \in A \subseteq M$ and a finite set $S \subset A$. We say that a is *certifiably separated from* b in A if there are $c, d \in A$ such that $f(c) = f(d) = 1$ and either $a \leq c < d \leq b$ or $b \leq c < d \leq a$. We say that a is *certifiably separated from* S in A if a is certifiably separated from every $b \in S$ in A. If $\mathcal{S} = \langle S_i : i \leq n\rangle$ is a collection of finite subsets of A, then we say that $\mathcal{S}$ is *certifiably separated* in A if whenever $i, j \leq n$, $i \neq j$ and $a \in S_i$, then a is certifiably separated from S_j in A.

Definition 4.4. Let g induce a separation of $\langle A, \widetilde{A}\rangle$ of length n. Then a *character* for this separation is a map $h : \{0, 1, \ldots, n-1\} \rightarrow \{0, 1, 2\}$.

When we have a set G of nested partial isomorphisms, we will sometimes shift our maps on intervals while modifying a partial isomorphism. The next definition will allow us to trace back the shifts.

Definition 4.5. Let $G = \langle g_i : i \leq k\rangle$ be a sequence of partial isomorphisms of A with $\widetilde{A}$ such that for all $i < k$, $\mathrm{dom}(g_i) \subseteq \mathrm{dom}(g_{i+1})$ and $\mathrm{rng}(g_i) \subseteq \mathrm{rng}(g_{i+1})$, and let the separation induced by g_i have length $j(i)$. Given $\ell < i \leq k$ and $j < j(i)$, we define $\overline{g}(i, j, \ell)$ to be the unique m such that $\widetilde{I}_{i,j}^{\widetilde{A}} \subseteq \widetilde{I}_{\ell,m}^{\widetilde{A}}$. Thus $\overline{g}$ matches nested intervals in the range of partial isomorphisms with their corresponding preimages.

We have now identified the individual functions and sets used by our constructions; M will remain fixed throughout this section. We will have to deal with finite sequences of partial isomorphisms and their accompanying characters, and will need these sequences to mesh well with each other. This will enable us both to enlarge $\widetilde{A}$ and also to extend g in a way which we know will be consistent with building an isomorphic copy of M. The next definition makes this discussion precise; the content of its clauses will be explained following the definition.

Definition 4.6. Let A and $\widetilde{A}$ be given such that $A \subseteq M$ and $\widetilde{A}$ is finite, and let $G = \langle g_i : i \leq k\rangle$ be a finite sequence of partial isomorphisms of A with $\widetilde{A}$. For every $i \leq k$, fix notation as above for the separation of $\langle A, \widetilde{A}\rangle$ induced by $\langle \mathrm{dom}(g_i), \mathrm{rng}(g_i)\rangle$, whose length is denoted by $j(i)$. Let $H = \langle h_i : i \leq k\rangle$ be a finite sequence of characters. The tuple $\langle A, \widetilde{A}, G, H\rangle$ is said to be an *environment* if for all $i < k$, $\mathrm{dom}(g_i) \subseteq \mathrm{dom}(g_{i+1})$ and $\mathrm{rng}(g_i) \subseteq \mathrm{rng}(g_{i+1})$. This environment is *linear* if $g_i \subseteq g_{i+1}$ for all $i < k$, and is *coherent* if the following conditions hold:

$$\forall i \leq k \forall j < j(i)(|I^A_{i,j}| \geq |\widetilde{I}^{\widetilde{A}}_{i,j}|). \tag{4.1}$$

$$\forall \ell < i \leq k \forall j < j(i)(h_\ell(\overline{g}(i,j,\ell)) = 1 \ \rightarrow \ h_i(j) = 1). \tag{4.2}$$

$$\forall i \leq k \forall j < j(i)(h_i(j) = 1 \ \rightarrow \ |I^A_{i,j}| = |\widetilde{I}^{\widetilde{A}}_{i,j}|). \tag{4.3}$$

$$\forall i < k \forall j < j(i+1)(h_i(\overline{g}(i+1,j,i)) = 1 \ \rightarrow \ |\widetilde{I}^{\widetilde{A}}_{i+1,j}| = |\widetilde{I}^{\widetilde{A}}_{i,\overline{g}(i+1,j,i)}|). \tag{4.4}$$

$$\forall \ell < i \leq k \forall j < j(i)(h_i(j) = 2 \ \rightarrow \ h_\ell(\overline{g}(i,j,\ell)) = 2). \tag{4.5}$$

$$\forall i \leq k \forall j < j(i)(|\{x \in [p^A_{i,j}, p^A_{i,j+1}] \cap A : f(x) = 1\}| \geq 2 \ \leftrightarrow \ h_i(j) = 2). \tag{4.6}$$

$$\forall i < k \forall j < j(i+1)(I^A_{i+1,j} \not\subseteq I^A_{i,\overline{g}(i+1,j,i)} \ \rightarrow \ h_{i+1}(j) \in \{1,2\}). \tag{4.7}$$

This environment is *M-coherent* if it is coherent and

$$\forall i \leq k \forall j < j(i)(h_i(j) = 1 \ \rightarrow \ I^M_{i,j} = I^A_{i,j}). \tag{4.8}$$

$$\forall i \leq k \forall j < j(i)(|\{x \in [p^A_{i,j}, p^A_{i,j+1}] \cap M : f(x) = 1\}| \geq 2 \ \rightarrow \\ |\{x \in [p^A_{i,j}, p^A_{i,j+1}] \cap A : f(x) = 1\}| \geq 2). \tag{4.9}$$

Condition (4.1) will ensure that we can extend a partial isomorphism of $A \subseteq M$ with $\widetilde{A}$ to a full isomorphism of M with some $\widetilde{M} \supseteq \widetilde{A}$. However, other requirements constrain the isomorphism by requiring that certain blocks of $\widetilde{A}$ remain blocks in $\widetilde{M}$, and so necessitate the addition of conditions (4.2)–(4.4) which ensure that we preserve the designation of blocks, allow designated blocks of $\widetilde{A}$ to be the images of blocks of A, and prevent designated blocks from growing in size. When we search for intervals in which to place new elements of $\widetilde{M}$, we will want all preimages of these intervals to have certifiably separated endpoints; conditions (4.5) and (4.6) keep track of certifiable separation and ensure that we only need to look at the last partial isomorphism to ensure certifiable separation for all partial isomorphisms. Condition (4.7) will ensure that when successive maps violate linearity, then we can precisely determine the character of each shifted interval. This will be necessary to ensure the co-hereditary nature of certifiable separation.

The expansion of $\widetilde{A}$ to $\widetilde{M}$ is under our control, and we can ensure that such expansions are coherent. However, we have no control over the expansion of A to M. Such expansions can destroy the satisfaction of (4.3); the satisfaction of the latter condition can only be ensured if the starting environment satisfies (4.8). (4.9) requires the declaration of the absence of certifiable separation to reflect the situation in M; as A grows, this will be the case.

We can effectively check for the coherence of a given environment, but can discover its M-coherence only in the limit. Thus we will have to devise new environments to satisfy requirements when we discover that the prior environment is no longer coherent. We arrange our constructions so that, for a given requirement, we will only have to do this finitely often along the true path of the construction.

We now define extensions of environments.

Definition 4.7. Let $E = \langle A, \widetilde{A}, G, H\rangle$ and $\widehat{E} = \langle C, \widetilde{C}, \widehat{G}, \widehat{H}\rangle$ be environments. We say that $\widehat{E}$ *extends* E (written as $E \subseteq \widehat{E}$) if $A \subseteq C$, $\widetilde{A} \subseteq \widetilde{C}$, and there are $\widetilde{G}$ and $\widetilde{H}$ such that $\widehat{G} = G^\frown\widetilde{G}$ and $\widehat{H} = H^\frown\widetilde{H}$.

The success of our constructions will depend on our ability to find M-coherent extensions of M-coherent environments with designated properties. We begin by analyzing the effect of adding new elements to A on the coherence or M-coherence of the environment $\langle A, \widetilde{A}, G, H\rangle$. We first show that the expansion of A has no effect on the M-coherence of the environment.

Lemma 4.8. *Suppose that $\langle A, \widetilde{A}, G, H\rangle$ is M-coherent and that $A \subseteq C \subseteq M$. Then $\langle C, \widetilde{A}, G, H\rangle$ is M-coherent.*

Proof. The only conditions from among (4.1)–(4.9) which do not follow immediately for C from the similar condition for A are (4.3) and (4.6). (4.3) for C in place of A follows from (4.3) and (4.8) for A; and (4.6) for C in place of A follows from (4.6) and (4.9) for A. □

When the starting environment $\langle A, \widetilde{A}, G, H\rangle$ is coherent, the addition of new elements to A can destroy the coherence of this environment in two ways. The first way is through the addition of elements to intervals on which a character h takes the value 1, thereby destroying a block and failing to satisfy (4.3). We will avoid this by not allowing h to take the value 1 unless forced to do so by an M-coherent subenvironment. The second way is through the discovery that an interval, to which h has assigned the value 0, now has certifiably separated endpoints. This can happen only finitely often, and requires us only to revise characters. The next definition describes the revisions to h, and is followed by a lemma covering the situation just described.

Definition 4.9. Let h and $\widehat{h}$ be characters with the same domain. We say that $h \preccurlyeq \widehat{h}$ if for all $j \in \mathrm{dom}(h)$, $h(j) \neq \widehat{h}(j)$ implies that $h(j) = 0$ and $\widehat{h}(j) = 2$.

Lemma 4.10. *Suppose that $\langle A, \widetilde{A}, G, H\rangle$ is M-coherent with $G = \langle g_i : i \leq k-1\rangle$ and $H = \langle h_i : i \leq k-1\rangle$, that $\langle A, \widetilde{A}, \widehat{G}, \widehat{H}\rangle$ is coherent with $\widehat{G} = \langle g_i : i \leq k\rangle$ and $\widehat{H} = \langle h_i : i \leq k\rangle$, that $g_{k-1} \subseteq g_k$, and that for all $j \in \mathrm{dom}(h_k)$, $h_k(j) = 1$ implies that $h_{k-1}(\overline{g}(k, j, k-1)) = 1$. Let $C \subseteq M$ be a finite extension of A. Then there is a character $\widehat{h}_k$ such that $h_k \preccurlyeq \widehat{h}_k$ and if $\widetilde{H} = H^\frown\langle\widehat{h}_k\rangle$, then $\langle C, \widetilde{A}, \widehat{G}, \widetilde{H}\rangle$ is coherent.*

Proof. As $g_{k-1} \subseteq g_k$ and $\langle A, \widetilde{A}, G, H\rangle$ is M-coherent, our limitations on where h_k takes the value 1 imply that $\langle A, \widetilde{A}, \widehat{G}, \widehat{H}\rangle$ can fail to be coherent only if the endpoints of the interval $I^C_{k,j}$ for some j such that $h_k(j) \neq 2$ are certifiably separated in C. Now $h_k(j) \neq 1$, else $h_{k-1}(\overline{g}(k, j, k-1)) = 1$, and as $I^A_{k,j} \subseteq I^A_{k-1,\overline{g}(k,j,k-1)}$, the entry of these new elements into the interval would destroy the M-coherence of $\langle A, \widetilde{A}, G, H\rangle$. Thus by redefining the character to take the value 2 for such j, we produce a character $\widehat{h}_k$ which is $\succcurlyeq h_k$ and satisfies the conclusion of the lemma. □

We now analyze when it is safe to expand $\widetilde{A}$, and where the new elements can be added. One location where elements can be safely added is in intervals $\widetilde{I}^{\widetilde{A}}_{k,j}$ corresponding to intervals $I^A_{k,j}$ whose endpoints are certifiably separated in A. We show that we can always find such an interval.

Lemma 4.11. *Let $\langle A, \widetilde{A}, G, H\rangle$ be M-coherent, with $|G| = k+1$, and assume that M is not condensationally finite. Then there is a $j < j(k)$ such that for all $i \le k$, the endpoints of $I^A_{i,\overline{g}(k,j,i)}$ are certifiably separated.*

Proof. As M is not condensationally finite, M has infinitely many maximal blocks, so as f is a choice function for M, $|\{x \in M : f(x) = 1\}| = \infty$. As any separation of M is finite, there is a $j < j(k)$ such that the endpoints of $I^A_{j,k}$ are certifiably separated. By (4.6), $h_k(j) = 2$, so by (4.5), $h_i(\overline{g}(k, j, i)) = 2$ for all $i \le k$. Now again by (4.6), the endpoints of $I^A_{i,\overline{g}(k,j,i)}$ are certifiably separated for all $i \le k$. □

When $h_k(j) = 2$, we can speed up the enumeration of M to permit the addition of any fixed finite number of elements into $\widetilde{I}^{\widetilde{A}}_{k,j}$. The next two lemmas show how this is done.

Lemma 4.12. *Let $\langle A, \widetilde{A}, G, H\rangle$ be M-coherent, with $|G| = k+1$, and fix $j < j(k)$ such that $h_k(j) = 2$. Then for any number m, there is a finite extension $C \subseteq M$ of A such that $\langle C, \widetilde{A}, G, H\rangle$ is M-coherent and for all $i \le k$,*

$$|I^C_{i,\overline{g}(k,j,i)}| \ge |\widetilde{I}^{\widetilde{A}}_{i,\overline{g}(k,j,i)}| + m. \tag{4.10}$$

Proof. As $h_k(j) = 2$, it follows from (4.5) that $h_i(\overline{g}(k, j, i)) = 2$ for all $i \le k$. By (4.6), the endpoints of $I^A_{i,\overline{g}(k,j,i)}$ are certifiably separated for all $i \le k$, so each of these intervals must be infinite. Thus we can speed up the enumeration of M to obtain a finite extension $C \subseteq M$ of A which satisfies (4.10). By Lemma 4.8, $\langle C, \widetilde{A}, G, H\rangle$ is M-coherent. □

Lemma 4.13. *Let $\langle A, \widetilde{A}, G, H\rangle$ be M-coherent, with $|G| = k+1$, and fix $j < j(k)$ such that (4.10) holds. Let $\widetilde{C}$ be a finite extension of $\widetilde{A}$ obtained by adding m-many new elements to $\widetilde{I}^{\widetilde{A}}_{k,j}$, arbitrarily ordered within that interval. Then $\langle A, \widetilde{C}, G, H\rangle$ is M-coherent.*

Proof. The only obstacle to M-coherence when adding new elements to $\widetilde{A}$ is the verification of (4.1). But this condition follows immediately from (4.10). □

Now suppose that we have an M-coherent environment, and want to extend the environment by appending a new partial isomorphism to G which adds one new pair, matching $c \in I^A_{k,j}$ with $\widetilde{c} \in \widetilde{I}^{\widetilde{A}}_{k,j}$. Our next lemma shows that this can be done if the following condition holds:

$$|[\widetilde{p}_{k,j}, \widetilde{c}]| \le |[p_{k,j}, c]| \ \& \ |[\widetilde{c}, \widetilde{p}_{k,j+1}]| \le |[c, p_{k,j+1}]|. \tag{4.11}$$

Lemma 4.14. *Let $\langle A, \widetilde{A}, G, H\rangle$ be M-coherent with $G = \langle g_i : i \le k\rangle$, and fix $j < j(k)$. Fix $c \in I^A_{k,j}$ and $\widetilde{c} \in \widetilde{I}^{\widetilde{A}}_{k,j}$ such that (4.11) holds. Define $g_{k+1} = g_k \cup \{\langle c, \widetilde{c}\rangle\}$. Then there is a character h_{k+1} such that if we set $\widehat{G} = \langle g_i : i \le k+1\rangle$ and $\widehat{H} = \langle h_i : i \le k+1\rangle$, then there is a finite extension $C \subseteq M$ of A such that $\langle C, \widetilde{A}, \widehat{G}, \widehat{H}\rangle$ is M-coherent.*

Proof. We first speed up the enumeration of M to obtain a finite extension $C \subseteq M$ of A such that $p_{k,j}$ and c are certifiably separated in C iff $p_{k,j}$ and c are certifiably separated in M and c and $p_{k,j+1}$ are certifiably separated in C iff c and $p_{k,j+1}$ are certifiably separated in M. Define $h_{k+1}(r) = h_k(r)$ if $r < j$ and $h_{k+1}(r) = h_k(r-1)$ if $r > j+1$. We set $h_{k+1}(j) = h_{k+1}(j+1) = h_k(j)$, unless $h_k(j) = 2$ and the endpoints of the corresponding interval are not certifiably separated in C; in the latter case, h_{k+1} takes the value 0 for that interval. (4.2)–(4.9) follow easily from the choice of C, the M-coherence of the starting environment and the definition of h_{k+1}, as does (4.1) for $i \leq k$. (4.1) for $i = k+1$ follows immediately from (4.11). □

The next lemma will be used to show that the range of the final isomorphism is $\widetilde{M}$.

Lemma 4.15. *Suppose that $\langle A, \widetilde{A}, G, H\rangle$ is M-coherent with $G = \langle g_i : i \leq k\rangle$, and fix $\widetilde{c} \in \widetilde{A}$. Then there is a finite extension $C \subseteq M$ of A, a partial isomorphism $g_{k+1} \supseteq g_k$ of C with $\widetilde{A}$ for which $\widetilde{c} \in \mathrm{rng}(g_{k+1})$ and a character h_{k+1} such that if $\widehat{G} = \langle g_i : i \leq k+1\rangle$ and $\widehat{H} = \langle h_i : i \leq k+1\rangle$, then $\langle C, \widetilde{A}, \widehat{G}, \widehat{H}\rangle$ is M-coherent.*

Proof. If $\widetilde{c} \in \mathrm{rng}(g_k)$, then the lemma is immediate for $g_{k+1} = g_k$ and $h_{k+1} = h_k$. Suppose that $\widetilde{c} \notin \mathrm{rng}(g_k)$. Fix $j < j(k)$ such that $\widetilde{c} \in \widetilde{I}^{\widetilde{A}}_{k,j}$. As $\widetilde{A}$ is finite, it follows from (4.1) that we can fix $c \in A$ satisfying (4.11). The lemma now follows from Lemma 4.14. □

The next lemma will be used to show that the domain of the final isomorphism is M.

Lemma 4.16. *Suppose that $\langle A, \widetilde{A}, G, H\rangle$ is M-coherent with $G = \langle g_i : i \leq k\rangle$, and fix $c \in A$. Then there are finite extensions $C \subseteq M$ of A and $\widetilde{C}$ of $\widetilde{A}$, a partial isomorphism g_{k+1} of C with $\widetilde{C}$ for which $c \in \mathrm{dom}(g_{k+1})$, and a character h_{k+1} such that if $\widehat{G} = \langle g_i : i \leq k+1\rangle$ and $\widehat{H} = \langle h_i : i \leq k+1\rangle$, then $\langle C, \widetilde{C}, \widehat{G}, \widehat{H}\rangle$ is M-coherent.*

Proof. If $c \in \mathrm{dom}(g_k)$, then the lemma is immediate for $g_{k+1} = g_k$ and $h_{k+1} = h_k$. Suppose that $c \notin \mathrm{dom}(g_k)$. Fix $j < j(k)$ such that $c \in I^A_{k,j}$. If $|\widetilde{I}^{\widetilde{A}}_{k,j}| > 2$, then we can fix $\widetilde{c} \in \widetilde{I}^{\widetilde{A}}_{k,j}$ satisfying (4.11). The lemma now follows from Lemma 4.14.

Suppose that $|\widetilde{I}^{\widetilde{A}}_{k,j}| = 2$. There are two cases, depending on the value of $h_k(j)$. In each case, we show that if the case is possible, then we can find a finite extension $C \subseteq M$ of A and a finite extension $\widetilde{C}$ of $\widetilde{A}$ such that $\langle C, \widetilde{C}, G, H\rangle$ is M-coherent and $|\widetilde{I}^{\widetilde{C}}_{k,j}| > 2$. This suffices, as we can then proceed as in the preceding paragraph to complete the proof.

Case 1: $h_k(j) \in \{0, 2\}$. Fix the greatest $\ell \leq k$, if any, such that $h_\ell(\overline{g}(k,j,\ell)) \neq 0$, and set $\ell = 0$ otherwise. By (4.7), $I^A_{k,j} \subseteq I^A_{i,\overline{g}(k,j,i)}$ for all i such that $\ell \leq i \leq k$. By our assumptions, $|\widetilde{I}^{\widetilde{A}}_{k,j}| < |I^A_{k,j}|$, so by (4.1) and the inclusion of intervals, it must be the case that $|\widetilde{I}^{\widetilde{A}}_{i,\overline{g}(k,j,i)}| < |I^A_{i,\overline{g}(k,j,i)}|$ for all i such that $\ell \leq i \leq k$. Note that $h_i(\overline{g}(k,j,\ell)) \neq 1$, else by (4.2), $h_k(j) = 1$. Thus $h_i(\overline{g}(k,j,\ell)) = 2$. Now by Lemma 4.12, we can speed up the enumeration of A to obtain C satisfying $|\widetilde{I}^{\widetilde{A}}_{i,\overline{g}(k,j,i)}| < |I^C_{i,\overline{g}(k,j,i)}|$ for all $i \leq \ell$ and such that $\langle C, \widetilde{A}, G, H\rangle$ is M-coherent. Thus (4.10) holds for C in place of A, $m = 1$, and all $i \leq k$. By Lemma 4.13, we can now add a new number $\widetilde{c}$ to $\widetilde{I}^{\widetilde{A}}_{k,j}$ to obtain $\widetilde{C}$, and $\langle C, \widetilde{C}, G, H\rangle$ will be M-coherent. We now see that $|\widetilde{I}^{\widetilde{C}}_{k,j}| > 2$, as required.

Case 2: $h_k(j) = 1$. By (4.8), $|\widetilde{I}^{\widetilde{A}}_{k,j}| = |I^{A}_{k,j}| \geq 3$, contrary to our assumption. Hence this case is impossible. □

We now consider the extensions of environments that will be needed to match blocks. We consider the setting of linear environments first, as it is easier. In this setting, we will start with a linear environment $\langle A, \widetilde{A}, G, H\rangle$ and a block $\widetilde{B} \subseteq \widetilde{I}^{\widetilde{A}}_{k,j}$ of $\widetilde{A}$, and will want to match this block with a block B of $I^{M}_{k,j}$. We will only be able to identify B in the limit, so will try, finitely often, to match $\widetilde{B}$ with unsuitable blocks before discovering the correct block B. While the unsuitable attempts are taking place, we will be expanding A and $\widetilde{A}$, so must take care that it is still possible to match B with $\widetilde{B}$ while preserving our ability to build M-coherent environments. Two types of obstacles are present in the setting of linear environments.

The first obstacle is that in adding elements to the model being built, we may add some between the endpoints of $\widetilde{B}$, thus destroying its existence as a block. Thus B will have the wrong size to be matched with $\widetilde{B}$. We prevent this by setting the character of all subintervals of B to take the value 1. The character 1 will be carried up to all coherent extensions, and will overcome this obstacle.

The second obstacle is that while we are acting for an unsuitable attempt, we may add new elements in the intervals to either side of the block $\widetilde{B}$. When shifting to a new attempt, the interval of M to which this side interval $\widetilde{D}$ is to be mapped may not be large enough to accommodate all the elements of $\widetilde{D}$. We solve this problem by requiring any block B chosen to be certifiably separated from all elements of the last separation of the environment whose counterparts are not in $\widetilde{B}$. This will allow us to speed up the enumeration of M so that $\widetilde{D}$ can be accommodated.

The character needed to overcome the first obstacle is not part of the starting environment. The purpose of the next definition is to establish a way of conveying the information carried by that character to new environments.

Definition 4.17. Let $E = \langle A, \widetilde{A}, G, H\rangle$ be coherent, with $G = \langle g_i : i \leq k\rangle$ and $H = \langle h_i : i \leq k\rangle$, fix $j < j(k)$, a block $\widetilde{B}$ of $\widetilde{I}^{\widetilde{A}}_{k,j}$, and a finite extension $\widetilde{C}$ of $\widetilde{A}$. Fix a finite coherent extension $\widetilde{E} = \langle C, \widetilde{C}, \widehat{G}, \widehat{H}\rangle$ of $\langle A, \widetilde{A}, G, H\rangle$, with $\widehat{G} \supseteq G^\frown\langle \widehat{g}_{k+1}\rangle$ and $\widehat{H} \supseteq H^\frown\langle \widehat{h}_{k+1}\rangle$. We say that $\widetilde{E}$ *respects* $\widetilde{B}$ if $\widetilde{B} \subseteq \mathrm{dom}(\widehat{g}_{k+1})$ and whenever both endpoints of $\widetilde{I}^{\widetilde{C}}_{k+1,m}$ are in $\widetilde{B}$, then $\widehat{h}_{k+1}(m) = 1$.

Lemma 4.18. *Fix notation as in Definition 4.17, and suppose that $\widetilde{E}$ respects $\widetilde{B}$. Let B be a block of $I^{M}_{k,j}$ of cardinality $|\widetilde{B}|$ with least element b_0 and greatest element b_1, and suppose that $p^{M}_{k,j}$ and b_0 are certifiably separated in M if $\widetilde{p}^{M}_{k,j} \notin \widetilde{B}$, and that $p^{M}_{k,j+1}$ and b_1 are certifiably separated in M if $\widetilde{p}^{M}_{k,j+1} \notin \widetilde{B}$. Suppose also that $\langle A, \widetilde{A}, G, H\rangle$ is M-coherent. Then there is a finite extension $\widehat{C} \subseteq M$ of C, a partial isomorphism g_{k+1} and a character h_{k+1} such that if $\widetilde{G} = G^\frown\langle g_{k+1}\rangle$ and $\widetilde{H} = H^\frown\langle h_{k+1}\rangle$ then $\widehat{E} = \langle \widehat{C}, \widetilde{C}, \widetilde{G}, \widetilde{H}\rangle$ is coherent and g_{k+1} maps B one-to-one onto $\widetilde{B}$ in an order-preserving fashion.*

Proof. As $|B| = |\widetilde{B}|$ and they lie in corresponding intervals of E, the only obstacles to constructing an appropriate g_{r+1} are the violation of (4.1) or the violation of the nature of $\widetilde{B}$ as a block. (Note that h_{r+1} can always be defined as in the proof of Lemma 4.14, modified to have characters for intervals with endpoints in $\widetilde{B}$ take the value 1.) As $\widetilde{E}$ respects $\widetilde{B}$ and is a coherent environment, it follows from (4.4) and (4.2) that $\widetilde{B}$ is a block of $\widetilde{C}$, so we can match B with $\widetilde{B}$ in a one-to-one, onto, order-preserving fashion. By the hypotheses on the certifiable separation of the endpoints of B from those of $I^M_{k,j}$, it follows from Lemma 4.12 that we can speed up the enumeration of M to find $\widehat{C}$ which provides sufficiently many elements in the intervals $[p^A_{k,j}, b_0] \cap \widehat{C}$ and $[b_1, p^A_{k,j+1}] \cap \widehat{C}$ to satisfy (4.1). The coherence of $\widetilde{E}$ now implies the coherence of $\widehat{E}$. □

We will have to match blocks on several intervals simultaneously. The next definition and lemma cover this situation.

Definition 4.19. Let $E = \langle A, \widetilde{A}, G, H\rangle$ be coherent with $G = \langle g_i : i \leq k\rangle$. Let $0 = p_{k,j_0} < p_{k,j_1} < \cdots < p_{k,j_r} = 1$ be a subsequence of the separating points of the separation of A induced by g_k, and for all $m < r$, let $E_m = \langle A, \widetilde{A}, G_m, H_m\rangle$ be an environment extending E such that $G_m = G^\frown\langle g_{k+1,m}\rangle$, and assume that $g_{k+1,m}(p_{k,j_m}) = p_{k,j_m}$ and $g_{k+1,m}(p_{k,j_{m+1}}) = p_{k,j_{m+1}}$ for all $m < r$. The *piecewise extension* $\widehat{E} = \langle A, \widetilde{A}, \widehat{G}, \widehat{H}\rangle$ of E *via* $\langle E_m : m < r\rangle$ is defined as follows. We set $\widehat{g}_{k+1}(x) = g_{k,m}(x)$ if $p_{k,j_m} \leq x \leq p_{k,j_{m+1}}$ and $x \in \mathrm{dom}(g_{k,m})$ (it follows from the preceding sentence that this map is well-defined). The character $\widehat{h}_{k+1}$ of an interval induced by $\mathrm{rng}(\widehat{g}_{k+1})$ is the character of that same interval as induced by $\mathrm{rng}(g_{k+1,m})$, if that interval is a subinterval of $[p_{k,j_m}, p_{k,j_{m+1}}]$. We set $\widehat{G} = G^\frown\langle \widehat{g}_{k+1}\rangle$ and $\widehat{H} = H^\frown\langle \widehat{h}_{k+1}\rangle$

Lemma 4.20. *Let E be coherent, and fix notation as in Definition 4.19. Suppose that each E_m is coherent. Then $\widehat{E}$ is coherent. Furthermore, if E is M-coherent and each E_m is M-coherent, then $\widehat{E}$ is M-coherent.*

Proof. The verification of (4.1)–(4.9) for $\widehat{E}$ follows routinely from the satisfaction of the corresponding condition by E and each E_j. We leave this verification to the reader. □

Lemmas 4.18 and 4.20 cover the extension of linear environments when blocks are to be matched. We will have to do the same when the environments fail to be linear, and in this case, will have to match finite sequences of blocks. There is a third obstacle to be overcome in this case, namely, as we shift, we will have to add new elements to $\widetilde{A}$ as images of elements of $\mathrm{dom}(g_k)$ which have had their counterpart in $\mathrm{rng}(g_k)$ matched with a new element. This obstacle is already taken care of by requiring that the blocks of M be certifiably separated from each other and from the endpoints of the union of intervals in which the shift takes place.

Lemma 4.21. *Let $E = \langle A, \widetilde{A}, G, H\rangle$ be M-coherent, with $G = \langle g_i : i \leq k\rangle$ and $H = \langle h_i : i \leq k\rangle$. Fix the smallest $\ell \leq k$ such that $g_\ell \subseteq g_i$ for all i such that $\ell \leq i \leq k$. Fix $j < j(\ell)$, a collection of pairwise disjoint blocks $\widetilde{\mathcal{B}}$ of $\widetilde{I}^{\widetilde{A}}_{\ell,j}$, and a collection of pairwise disjoint blocks $\mathcal{B}$ of $I^A_{\ell,j}$. Let $\widehat{E} = \langle C, \widetilde{C}, \widehat{G}, \widehat{H}\rangle$ be a finite coherent extension of E which respects all blocks in $\widetilde{\mathcal{B}}$.*

Assume that:

(i) Each subinterval $\widetilde{I}^{\widetilde{A}}_{k,r}$ of $\widetilde{I}^{\widetilde{A}}_{\ell,j}$ whose character is 1 is a subblock of some block in $\widetilde{\mathcal{B}}$;

(ii) Each element of $\mathrm{rng}(g_k) \cap \widetilde{I}^{\widetilde{A}}_{\ell,j}$ is an element of some block in $\widetilde{\mathcal{B}}$;

(iii) If $\ell \leq i \leq k$ and $c, d \in \mathrm{dom}(g_i) \cap I^A_{\ell,j}$ are not certifiably separated, then $g_i(c)$ and $g_i(d)$ are in the same block of $\widetilde{\mathcal{B}}$;

(iv) The blocks in $\mathcal{B}$ are pairwise certifiably separated;

(v) $|\mathcal{B}| = |\widetilde{\mathcal{B}}| - \in$;

(vi) Each element of $\mathcal{B}$ has cardinality greater than or equal to the cardinality of the largest block in $\widetilde{\mathcal{B}}$; and

(vii) Each element of $\mathcal{B}$ is certifiably separated from all elements of $\mathrm{dom}(g_k)$.

Then there is a finite extension $D \subseteq M$ of C and a partial isomorphism g_{k+1} which agrees with g_k except on the elements in the interior of $I^A_{\ell,j}$ and matches an initial segment of the ith block of $\mathcal{B}$ with the $(i+1)$st block of $\widetilde{\mathcal{B}}$ (thus the first and last blocks of $\widetilde{\mathcal{B}}$ are not matched with blocks of $\mathcal{B}$) and a corresponding character h_{k+1} such that if $\widetilde{G} = G^\frown\langle g_{r+1}\rangle$ and $\widetilde{H} = H^\frown\langle h_{r+1}\rangle$ then $\widehat{E} = \langle \widehat{C}, \widetilde{C}, \widetilde{G}, \widetilde{H}\rangle$ is coherent.

Proof. By (v) and (vi) and as $\widehat{E}$ respects all blocks in $\widetilde{\mathcal{B}}$, we can map the elements in the blocks of $\mathcal{B}$ to the blocks of $\widetilde{\mathcal{B}}$ as described. By (iii), (iv) and (vii), we can speed up the enumeration of M to obtain D in which the intervals between blocks of $\mathcal{B}$ and those separating the blocks of $\mathcal{B}$ from the endpoints of $I^A_{\ell,j}$ are large enough to accommodate the shifting of elements of $\widetilde{C}$ into the corresponding intervals of $\widetilde{I}^{\widetilde{A}}_{\ell,j}$ while satisfying (4.1), and also allowing g^{-1}_{k+1} and g^{-1}_k to agree on the first and last blocks of $\widetilde{\mathcal{B}}$, and h_{k+1} to agree with h_k on the corresponding intervals. We also assign character 1 to the interval of consecutive elements of each block of $\widetilde{\mathcal{B}}$. By (i) and (ii), this extension preserves all intervals of character 1 without enlarging them, so all conditions involving character 1 are satisfied.

Suppose that $\widetilde{c} = g_{k+1}(c)$ and $\widetilde{d} = g_{k+1}(d)$ are consecutive elements of $\mathrm{rng}(g_{k+1})$ and that c and d are certifiably separated in D. Fix i such that $\ell \leq i \leq k$. Let $\widehat{c}$ be the greatest element of $\mathrm{dom}(g_i)$ which is $\leq c$, and let $\widehat{d}$ be the smallest element of $\mathrm{dom}(g_i)$ which is $\geq d$. Then by (iii) and the M-coherence of E, $\widehat{c}$ and $\widehat{d}$ must be certifiably separated in A. Hence by (4.5), if $\ell \leq i \leq k$, then the conditions on character 2 propagate down from $k+1$ to i. And if $i < \ell$, then as $\overline{g}(k+1, r, i) = \overline{g}(\ell, \overline{g}(k+1, r, \ell), i)$, the conditions on character 2 propagate down from $k+1$ to ℓ and then from ℓ to i. We now see that h_{k+1} can be defined to satisfy the conclusion of the lemma. □

We now describe how some of these lemmas are used to satisfy R_n. We begin with the simpler situation, that encountered in the Large Block Construction. We will be given intervals $[a, b]$ of A and $[\widetilde{a}, \widetilde{b}]$ of $\widetilde{A}$ and a finite n-block $\widetilde{B} \subseteq [\widetilde{a}, \widetilde{b}] \cap \widetilde{A}$ for some n, and will want to find an n-block B of $[a, b] \cap M$ to match with $\widetilde{B}$. If $C \subset M$ is a finite extension of A and B is a block of C, then we can certify that B is a block of M only in a Π^0_1 way. Thus we may encounter finitely many false candidates for B and try, for a while, to match them to $\widetilde{B}$, before identifying a true candidate for B which we adopt as our final candidate. This will require us to use an algorithm γ to generate the choices of candidates.

We digress to consider an example of such a γ. Let $\widetilde{B} = [\widetilde{c}, \widetilde{d}] \cap \widetilde{A}$. γ may adopt the strategy of trying to match the m-block $[\widetilde{a}, \widetilde{d}] \cap \widetilde{A}$ with an initial segment of $[a, b] \cap M$. This will be a problem if the maximal initial block of $[a, b] \cap M$ is a $< m$-block. In this case, there must eventually be a decision to terminate the search, as it cannot succeed. However, if we know that the maximal initial block of $[a, b] \cap M$ is an ω-block, then this strategy will be successful. Hence we will need another algorithm β which will advise us infinitely often to cancel searches on intervals which do not have a maximal initial ω-block, and will never advise cancellation of a search begun at a sufficiently large stage on some interval which has a maximal initial ω-block.

It may not always be possible to devise such a β, even if we know that there is always an interval with an initial ω-block. One situation in which we can devise such a β is when the initial maximal block of M is an ω-block, and M has a computable choice function f which chooses 0. In this case, β will advise us to work only in the first closed interval of C containing the second point of C chosen; this interval will always have an initial ω-block, and can effectively be determined at each stage.

Another potential obstacle needs to be addressed. We may try to extend $\widetilde{A}$ when we are working with an incorrect candidate. While doing so, if we allow new elements to enter $[\widetilde{a}, \widetilde{d}]$, then the true candidate, when finally found, may no longer be usable, as it may be incompatible with building an isomorphic copy of M. Thus if a search has begun and not yet been canceled, we must impose restrictions preventing the entry of new elements into designated intervals of $\widetilde{A}$. Such restrictions must be compatible with building an isomorphic copy of M, and must also ensure that once γ settles on a final candidate, then that candidate can be used. Thus we will require that γ also produce a number u which will be used to determine whether or not we want to extend the block-matching to match $[a, d]$ with $[\widetilde{a}, \widetilde{d}]$.

Once we decide to begin a search at stage s, the algorithms will ensure that we will either successfully complete that search or that the search will be canceled. In our constructions, we will have approximations M^s to M which will be generated by a computable enumeration of M. We will speed up the construction until the search either produces a candidate or is canceled, thus producing some M^t for $t > s$ in place of M^s. In such situations, we will reset $M^s = M^t$; there is no harm in making such a notational change.

We now describe the conditions required of β and γ to ensure the success of the Large Block Construction. The corresponding property will be a property of an order-type α, but will depend on α having a computable model M with a computable choice function. The algorithms will have a parameters P, describing the property imposed on the computable linear ordering M which ensures that β and γ will succeed (in our example, P is the property that the interval in which the search is taking place has an initial ω-block). β will have, as arguments, $a < b \in M$ and a stage s. γ will depend on an additional argument, the size n of the block being sought. Within a construction, n will have to be determined effectively not only from a, b and s, but also from the order-type α, the starting environment $E = \langle A, \widetilde{A}, G, H\rangle$, and a block $\widetilde{B}$ of some interval $I^{\widetilde{A}}$, so will be denoted as $n(a, b, E, \widetilde{B}, s)$.

Definition 4.22. Fix a computable order-type α. Fix $a < b \in M$, integers s and n, and a property P of sub-order-types of α. The *large block property* (abbreviated as LBP) for α postulates the existence of a computable model M of α with a computable choice function f and algorithms $\beta = \beta(a, b, s)$ and $\gamma = \gamma(a, b, n, s)$ satisfying the following conditions:

(i) (Correctness of β) If $[a, b] \cap M$ fails to have property P, then $\beta(a, b, s)$ advises cancellation of all searches on (a, b) at infinitely many stages s; and if $[a, b] \cap M$ has property P, then $\beta(a, b, s)$ advises continuation of all searches on (a, b) at all sufficiently large stages s.

(ii) (Adequacy of β) Every finite separation M, has at least one pair of consecutive elements $a < b$ such that $[a, b] \cap M$ has property P. Furthermore, if $[a, b]$ has property P, then (a, b) contains at least two maximal blocks of M.

(iii) (Local Honesty of γ) If $[a, b] \cap M$ has property P, then for all sufficiently large s, $\gamma(a, b, n, s)$ produces a a number $u \in \{0, 1\}$ and an n-block B of $[a, b] \cap M^s$ which is certifiably separated from b, is certifiably separated from a if $u = 0$, and contains a if $u = 1$.

(iv) (Uniformity) Suppose that $[a, b]$ has property P. For every stage s, let B^s and u^s be the objects (if any) generated by $\gamma(a, b, n, s)$ at stage s. Then there is a stage s_0 such that for all $t \geq s_0$, $B^t = B^{s_0}$. Furthermore, for any $r < t$ such that u^r and u^t are both defined, $u^t = u^r$.

We now exhibit two classes of order-types possessing models which have LBP. We will apply the Large Block Construction to each of these order-types in the next section.

Lemma 4.23. *Let α be a computable order-type with infinitely many maximal blocks such that for every n and every infinite interval of α, there is a $\geq n$-block in that interval of α. Assume that α has a computable model M with a computable choice function f. Then α has LBP.*

Proof. We first define the algorithm β for α. Fix an interval $[a, b]$ of M and an integer s. $\beta(a, b, s)$ will advise a search on $[a, b]$ iff

$$|\{x \in [a, b] \cap M^s : f(x) = 1\}| \geq 4. \tag{4.12}$$

Let $\widehat{a}$ and $\widehat{b}$ be the second and third points designated by f in (4.12) at the first stage at which this condition holds. As f is a choice function, (4.12) ensures that

$$|[a, \widehat{a}] \cap M| = \infty \ \& \ |[\widehat{a}, \widehat{b}] \cap M| = \infty \ \& \ |[\widehat{b}, b] \cap M| = \infty. \tag{4.13}$$

Definition 4.22(i) is immediate, and Definition 4.22(ii) follows easily from our assumption that α has infinitely many maximal blocks and that f is a choice function, since every separation has only finitely many intervals.

Fix a computable ordering $\mathcal{O}$ of all finite sequences of integers. $\gamma(a, b, n, s)$ will produce objects only if $\beta(a, b, s)$ advises a search on $[a, b]$. In this situation, the block B^s produced by γ will be the first element of $\mathcal{O}$ which is an n-block of $[\widehat{a}, \widehat{b}] \cap M^s$ if such a block exists; and if this block exists, then $\gamma(a, b, n, s)$ will set $u^s = 0$. It follows easily from our hypotheses, (4.12) and (4.13) that Definition 4.22(iii),(iv) are satisfied. □

Lemma 4.24. *Let $\alpha = \omega + \delta$ be a computable order-type such that δ does not have a least element and for all $a \in \delta$, $(-\infty, a]$ has infinitely many blocks. Assume that α has a computable model M with a computable choice function f. Then α has LBP.*

Proof. We first define the algorithm β for α. Fix an interval $[a,b]$ of M and an integer s. $\beta(a,b,s)$ will advise a search on $[a,b]$ iff the following conditions hold:

$$|\{x \in [0,a] \cap M^s : f(x) = 1\}| = 1; \tag{4.14}$$

$$|\{x \in [0,b] \cap M^s : f(x) = 1\}| \geq 2. \tag{4.15}$$

It follows from our assumptions that f is a choice function for M, $f(0) = f(1) = 0$, and M has infinitely many maximal blocks that any separation of A will provide us with an interval which satisfies these conditions. Definition 4.22(i) and (ii) now follow easily.

Fix a computable ordering $\mathcal{O}$ of all finite sequences of integers. $\gamma(a,b,n,s)$ will produce objects only if $\beta(a,b,s)$ advises a search on $[a,b]$. In this situation, the block B^s produced by γ will be the first element of $\mathcal{O}$ of the form $[a,c]$ which is an n-block of $[a,b] \cap M^s$ if such a block exists; and if this block exists, then $\gamma(a,b,n,s)$ will set $u^s = 1$. It follows easily from our hypotheses that the first maximal block of $[a,b]$ has order-type ω, and from (4.14) and (4.15), that c is certifiably separated from b, so Definition 4.22(iii) and (iv) are satisfied. □

Lemmas 4.23 and 4.24 demonstrate how a choice function f can be used to make separate attempts to match a block of $\widetilde{M}$ with each block within a given interval of M, while preserving our ability to build an isomorphism. In particular, an interval whose endpoints are certifiably separated must be infinite, so provides the flexibility to move points around. This will be important as well in the Small Block Construction.

The algorithms for the Small Block Construction are required to have similar properties to those listed in Definition 4.22, but require an additional level of approximation. We will be searching for a finite sequence of m-many blocks for some m to match with those in $\widetilde{\mathcal{B}}$ (which replaces $\widetilde{B}$), and m will vary with the environment. We will require β to depend on a block size n, both β and γ to depend on the additional variable m, and γ to depend also on a finite subset U of $[a,b]$ of points from which the blocks being sought are to be certifiably separated; both algorithms will be required to satisfy an additional limit condition.

The success of the Small Block Construction will depend not only on the small block property, but also on additional properties ensured by the order-types to which it is applied, and, in addition, the limit condition will have a different nature on each; thus rather than taking a limit on m, we will take a limit on $q(n)$, a number which will be effectively defined based on the order type. In one case, we will set $q(n) = r$ for some fixed number r, and in the other, we will set $q(n) = n$.

Definition 4.25. Fix a computable order-type α. Fix $a < b \in M$, integers s, n and m, and a set U such that $a, b \in U$. Then $[a,b]$ will have property P iff for all n, $[a,b]$ contains an infinite set of pairwise certifiably separated $q(n)$-blocks which are also certifiably separated from all elements of U. $[a,b]$ will have property Q iff for all n, $[a,b]$ contains a finite maximal $< q(n)$-block which is certifiably separated from all elements of U. The *small block property* for α (abbreviated as SBP) postulates the existence of a computable model M of α with a computable choice function f and algorithms $\beta = \beta(a,b,n,m,s)$ and $\gamma = \gamma(a,b,U,n,m,s)$ satisfying the following conditions:

(i) (Correctness of β) If $[a,b] \cap M$ fails to have property P, then there is an m such that for all sufficiently large n, $\beta(a,b,n,m,s)$ advises cancellation of all searches on (a, b) at infinitely many stages s; and if $[a, b] \cap M$ has property P, then for all m and n, $\beta(a, b, n, m, s)$ advises continuation of all searches on (a, b) at all sufficiently large stages s.

(ii) (Adequacy of β) Every finite separation of M, has at least one pair of consecutive elements $a < b$ such that $[a, b] \cap M$ has properties P and Q.

(iii) (Local Honesty of γ) If $\beta(a, b, n, m, s)$ advises continuation of searches at all sufficiently large s, then for all sufficiently large s, $\gamma(a, b, U, n, m, s)$ produces a set $\mathcal{B}$ of m-many pairwise certifiably separated $q(n)$-blocks of $[a, b] \cap M^s$ which are also certifiably separated from all elements of U.

(iv) (Uniformity) Suppose that $[a, b]$ has property P. For every stage s, let $\mathcal{B}^s$ be the set of blocks (if any) produced by $\gamma(a, b, U, n, m, s)$ at stage s. Then for each m and n, there is a stage s_0 such that for all $t \geq s_0$, $\mathcal{B}^t = \mathcal{B}^{s_0}$.

(v) (Faithfulness of Large Blocks) If $q(n) < n$ for some n, then all maximal blocks of M have cardinality $\leq q(0)$.

The Large Block Construction uses linear environments, so needs only to deal with one block at a time and to extend the last partial isomorphism. The Small Block Construction requires us to search for maximal blocks of a given size, and we only see how to do this using non-linear environments. The certifiable separation required of the large blocks will enable us to construct the environments directly within the constructions. The next two lemmas provide classes of order-types which have the small block property.

Lemma 4.26. *Let α be v-bounded but not finite-constant. Assume that α has a computable model M with a computable choice function f. Then α has SBP.*

Proof. By Lemma 2.2, we may assume that every finite separation of α has a v-bounded interval which is neither $(v-1)$-bounded nor v-constant. Fix the largest $k < v$ such that every finite separation of α has a $\langle k, v\rangle$-bounded interval which is not constant. By Lemma 2.2, we may assume that α is $\langle k, v\rangle$-bounded. A large block will be a v-block, so we set $q(n) = v$ for all n. We must define β and γ. We fix an ordering $\mathcal{O}$ of all finite sequences, each of whose elements is a finite sequence of integers. $\beta(a, b, n, m, s)$ will have a counter which is set to 0 for $s = 0$, and occasionally incremented. Suppose that the counter is set at r at stage s. β will advise continuation of the search if one of the first r elements of $\mathcal{O}$ provides a sequence of m-many pairwise certifiably separated v-blocks of $[a, b]$ which are also certifiably separated from all elements of U; otherwise, $\beta(a, b, n, m, s)$ advises cancellation of the search, and increments the counter to $r+1$ for stage $s+1$. When β advises continuation of the search, $\gamma(a, b, U, n, m, s)$ selects the first sequence of $\mathcal{O}$ consisting of m-many certifiably pairwise separated v-blocks of $[a, b]$ which are also certifiably separated from all elements of U.

As α is $\langle u, v\rangle$-bounded, any two maximal blocks of α are $\leq v$-blocks and are certifiably separated in M. The lemma now follows easily from the properties of α. □

Lemma 4.27. *Let α be non-finite-bounded with no completely infinite closed subinterval, and assume that α is not finite-constant. Assume that α has a computable model M with a computable choice function f. Then α has SBP.*

Proof. A large block will be an n-block, so we set $q(n) = n$ for all n. We must define β and γ. We fix an ordering $\mathcal{O}$ of all finite sequences, each of whose elements is a finite sequence of integers. $\beta(a, b, n, m, s)$ will have a counter which is set to 0 for $s = 0$, and occasionally incremented. Suppose that the counter is set at r at stage s. β will advise continuation of the search if one of the first r elements of $\mathcal{O}$ provides a sequence of m-many pairwise certifiably separated n-blocks of $[a, b]$ which are also certifiably separated from all elements of U; otherwise, $\beta(a, b, n, m, s)$ advises cancellation of the search, and increments the counter to $r + 1$ for stage $s + 1$. When β advises continuation of the search, $\gamma(a, b, U, n, m, s)$ selects the first sequence of $\mathcal{O}$ consisting of m-many pairwise certifiably separated n-blocks of $[a, b]$ which are also certifiably separated from all elements of U.

All intervals selected by the Σ_2-outcome of β have property P, and by the hypotheses on α, there will always be such an interval. Furthermore, as α has no completely infinite closed subinterval, each such interval will also have property Q. The lemma now follows easily. $\square$

We are now ready to present the constructions.

5. The Large Block Construction

Requirements will be assigned to nodes of a tree of strategies, and at stage s of the construction, a node σ will be chosen and an attempt will be made to satisfy the requirement assigned to σ. The attempt will be made taking the *environment* E^s_σ of σ at stage s into account. We are now ready to state the Large Block Theorem.

Theorem 5.1. (Large Block Theorem) *If there is a computable model of α which satisfies LBP and has infinitely many $\geq n$-blocks for all n, then α does not admit computable choice sets.*

Proof. We first present the basic modules for satisfying requirements. The construction will be a Δ^0_2 construction, so higher priority requirements will provide us with an environment $E^{s-1}_\sigma = \langle M^{s-1}, \widetilde{M}^{s-1}, G^{s-1}_\sigma, H^{s-1}_\sigma \rangle$ for each stage s and each node σ of our tree of strategies which is not initialized at the beginning of stage s, and $G^t_\sigma = G^r_\sigma$ and $H^t_\sigma = H^r_\sigma$ for all sufficiently large r and t. We will always let g^s_σ denote the last partial isomorphism in G^s_σ, and let h^s_σ denote the last character in H^s_σ. Let $G^t_\sigma = \langle g^t_{\sigma,i} : i \leq k \rangle$ and $H^t_\sigma = \langle h^t_{\sigma,i} : i \leq k \rangle$ for all t. We will write $I^s_{\sigma,j}$ in place of $I^{M^s}_{\sigma,j}$ and $\widetilde{I}^s_{\sigma,j}$ in place of $\widetilde{I}^{\widetilde{M}^s}_{\sigma,j}$ We assume, by induction, that all environments are coherent.

The first requirement is that $n \in \text{dom}(g)$. As our environments are coherent, it will be easy to extend the finite partial isomorphism map of the environment to a map whose domain includes n while preserving coherence and M-coherence.

The basic module for $\mathbf{P_n}$: Let this module begin at σ. σ will have a single outcome, $\sigma^\frown\langle 0 \rangle$ on the tree of strategies, which will always be followed. There are three cases.

Case 1: $n \in \text{dom}(g^s_\sigma)$. Define $E^s_{\sigma^\frown\langle 0 \rangle} = E^s_\sigma$.

Case 2: Case 1 does not apply and $n \in \text{dom}(g^{s-1}_{\sigma^\frown\langle 0 \rangle})$. If the environment $\langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\sigma^\frown\langle 0 \rangle}, H^{s-1}_{\sigma^\frown\langle 0 \rangle} \rangle$ is coherent, define

$$E^s_{\sigma^\frown\langle 0 \rangle} = \langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\sigma^\frown\langle 0 \rangle}, H^{s-1}_{\sigma^\frown\langle 0 \rangle} \rangle.$$

Otherwise, by Lemma 4.10, there will be a character $h^s_{\sigma^\frown\langle 0\rangle} \succcurlyeq h^{s-1}_{\sigma^\frown\langle 0\rangle}$ such that if we let $H^s_{\sigma^\frown\langle 0\rangle}$ be the sequence of characters obtained from $H^{s-1}_{\sigma^\frown\langle 0\rangle}$ by replacing $h^{s-1}_{\sigma^\frown\langle 0\rangle}$ with $h^s_{\sigma^\frown\langle 0\rangle}$, then $\langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\sigma^\frown\langle 0\rangle}, H^s_{\sigma^\frown\langle 0\rangle}\rangle$ will be coherent. We now set

$$E^s_{\sigma^\frown\langle 0\rangle} = \langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\sigma^\frown\langle 0\rangle}, H^s_{\sigma^\frown\langle 0\rangle}\rangle.$$

Case 3: Otherwise. Define $E^s_{\sigma^\frown\langle 0\rangle}$ extending $E^s_\sigma \cup \{\langle n, \widetilde{c}\rangle\}$ as in Lemma 4.16.

We now turn to ensuring that each n is in $\mathrm{rng}(g)$. As our environments are coherent, this will also be easy to do.

The basic module for $\mathbf{Q_n}$: Let this module begin at σ. σ will have a single outcome, $\sigma^\frown\langle 0\rangle$ on the tree of strategies, which will always be followed. There are three cases.

Case 1: $n \in \mathrm{rng}(g^s_\sigma)$. Define $E^s_{\sigma^\frown\langle 0\rangle} = E^s_\sigma$.

Case 2: Case 1 does not apply and $n \in \mathrm{rng}(g^{s-1}_{\sigma^\frown\langle 0\rangle})$. If the environment $\langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\sigma^\frown\langle 0\rangle}, H^{s-1}_{\sigma^\frown\langle 0\rangle}\rangle$ is coherent, define

$$E^s_{\sigma^\frown\langle 0\rangle} = \langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\sigma^\frown\langle 0\rangle}, H^{s-1}_{\sigma^\frown\langle 0\rangle}\rangle.$$

Otherwise, by Lemma 4.10, there will be a character $h^s_{\sigma^\frown\langle 0\rangle} \succcurlyeq h^{s-1}_{\sigma^\frown\langle 0\rangle}$ such that if we let $H^s_{\sigma^\frown\langle 0\rangle}$ be the sequence of characters obtained from $H^{s-1}_{\sigma^\frown\langle 0\rangle}$ by replacing $h^{s-1}_{\sigma^\frown\langle 0\rangle}$ with $h^s_{\sigma^\frown\langle 0\rangle}$, then $\langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\sigma^\frown\langle 0\rangle}, H^s_{\sigma^\frown\langle 0\rangle}\rangle$ will be coherent. We now set

$$E^s_{\sigma^\frown\langle 0\rangle} = \langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\sigma^\frown\langle 0\rangle}, H^s_{\sigma^\frown\langle 0\rangle}\rangle.$$

Case 3: Otherwise. First suppose that $n \notin \widetilde{M}^s$. By Lemma 4.13, we can speed up the enumeration of M and either find a t and a finite extension $\widetilde{A} \supset \widetilde{M}^s$ such that $\widehat{E}^t_\sigma = \langle M^t, \widetilde{A}, G^s_\sigma, H^s_\sigma\rangle$ is coherent and $n \in \widetilde{A}$, or find a t at which we can certify that E^t_σ is not M-coherent. In the latter case, we restart stage s at this new stage.

Suppose that $\widehat{E}^t_\sigma$ is found. Without loss of generality, we assume that $t = s$. We now define $E^s_{\sigma^\frown\langle 0\rangle}$ extending $\widehat{E}^t_\sigma$ as in Lemma 4.15.

We now turn to the satisfaction of R_n. As $\mathcal{M}$ has LBP, we can fix the algorithms β and γ and the block-size algorithm n provided by LBP in Definition 4.22. The idea will be to find an interval of $\widetilde{M}$ on which ϕ_n chooses at least two numbers, and use these numbers as the endpoints of a finite block which will be matched with a block of M. LBP will tell us when it is safe to try to find a matching block of M, and will ensure that if ϕ_n is total and a choice function, then we will eventually find such a matching block. Let $j(\sigma, s) = |\,\mathrm{dom}(g^s_\sigma)| - 1$. We will assume by induction that at each stage s, there is at most one current attack at stage s in each interval $I^s_{\sigma,j}$.

The basic module for $\mathbf{R_n}$: We work under the assumption that $\mathcal{M}$ has LBP. Let this module begin at σ. The possible outcomes for nodes are 0 and 1, with priority set so that outcome 0 has higher priority than outcome 1. There are several steps.

Step 1: (*Initiate Simultaneous Attacks*) There is no current attack for σ. (An attack for σ is *current* at stage s if it was initiated at some stage $r < s$ and not canceled at any stage t such that $r < t \leq s$.) For each $j < j(\sigma, s)$, let $I^s_{\sigma,j} = (a^s_{\sigma,j}, b^s_{\sigma,j})$ and $\widetilde{I}^s_{\sigma,j} = (\widetilde{a}^s_{\sigma,j}, \widetilde{b}^s_{\sigma,j})$. For each $j < j(\sigma, s)$, such that $\beta(a^s_{\sigma,j}, b^s_{\sigma,j}, s)$ advises that a search should take place on $I^s_{\sigma,j}$ at stage s, fix the least pair $\langle \widetilde{c}_j, \widetilde{d}_j \rangle$ of distinct elements of $\widetilde{M}^s$ (under some fixed computable ω-ordering of pairs of natural numbers) such that $\phi^s_n(\widetilde{c}_j) = \phi^s_n(\widetilde{d}_j) = 1$ and $\widetilde{c}_j < \widetilde{d}_j$ both lie in $\widetilde{I}^{s-1}_{\sigma,j}$. (We do this separately for each interval, and select a pair only when such a pair exists, a condition which can be effectively decided). An *attack* for σ *on* $I^s_{\sigma,j}$ *using* $\widetilde{B}_j$ is *initiated* at stage s for each j for which $\widetilde{c}_j$ and $\widetilde{d}_j$ exist, where $\widetilde{B}_j = [\widetilde{c}_j, \widetilde{d}_j] \cap \widetilde{M}^s$ if $u = 0$ and $\widetilde{B}_j = [\widetilde{a}^s_{\sigma,j}, \widetilde{d}_j] \cap \widetilde{M}^s$ if $u = 1$. If there are no attacks to initiate, then we follow outcome $\sigma^\frown\langle 1 \rangle$ and set $E^s_{\sigma^\frown\langle 1 \rangle} = E^s_\sigma$; $\sigma^\frown\langle 1 \rangle$ becomes a *terminal node* of the module. Suppose that attacks are initiated. We define the parameter $n(\sigma, j, s)$ for γ by $n(\sigma, j, s) = |\widetilde{B}_j|$. We now follow outcome $\tau = \sigma^\frown\langle 0 \rangle$ and go to Step 2.

Step 2: We speed up the enumeration of M until we have found a stage t such that, for each $j < j(\sigma, s)$ for which there was a current attack for σ on $I^s_{\sigma,j}$ at the beginning of stage s, either $\beta(a^s_{\sigma,j}, b^s_{\sigma,j}, s)$ advises cancellation of searches on $I^s_{\sigma,j}$ at some stage r such that $s \leq r \leq t$, or $\gamma(a^s_{\sigma,j}, b^s_{\sigma,j}, n(\sigma, j, s), s)$ produces a block and an integer for that search at stage t. Without loss of generality, we may assume that $t = s$. We *cancel* all attacks for which β advises cancellation. There are two substeps.

Substep 2.1: (*Cancel Attacks*) All current attacks for σ have been canceled. We follow outcome $\rho = \tau^\frown\langle 1 \rangle$ and set $E^s_{\tau^\frown\langle 1 \rangle} = E^s_\sigma$. $\tau^\frown\langle 1 \rangle$ becomes a *terminal node* of the module.

Substep 2.2: (*Extend the Environment*) Otherwise. Then $\gamma(a^s_{\sigma,j}, b^s_{\sigma,j}, n(\sigma, j, s), s)$ identifies a block B_j as in Definition 4.22(iii) for each j for which there is a current attack for σ on $I^s_{\sigma,j}$. If E^s_σ does not respect some $\widetilde{B}_j$, then the construction halts (we will show that this does not occur). Otherwise, we speed up the enumeration of M (we can assume that the resulting stage is s to get an environment $E^s_{\sigma,j}$ as produced by Lemma 4.18 (the first environment for that lemma is that at the stage where the attack was initiated, and the second is that at the end of stage $s - 1$)) for each j for which there is a current attack for σ on $I^s_{\sigma,j}$, and we set $E^s_{\sigma,j} = E^s_\sigma$ for all other $j < j(\sigma, s)$. Follow outcome $\sigma^\frown\langle 0 \rangle$ which becomes a *terminal node* of the module. Let $E^s_{\sigma^\frown\langle 0 \rangle}$ be the piecewise extension of E^s_σ for $\langle E^s_{\sigma,j} : j < j(\sigma, s) \rangle$.

We now define our tree of strategies in the usual way, specify a computable ordering of all requirements, and assign requirements to the trees, in order, as specified in our basic modules. The construction at stage s then follows the description of the basic modules, proceeding from node to node, initializing nodes to the right of the current path, and stopping and specifying the *current path* when it either specifies an outcome for a node which is in the initialized state, or does not specify an outcome for the node. Let Λ^s be the current path at stage s, and let Λ be the *true path* for the construction, i.e. the path, each of whose nodes is contained in Λ^s for cofinitely many s. Λ^s is *uninitialized* at stage $s + 1$.

We must now prove that $|\Lambda| = \infty$ and that all requirements are satisfied. The proof will use the next three lemmas, one for each type of requirement.

We begin with the satisfaction of P_n.

Lemma 5.2. *Suppose that s and σ are given such that P_n is assigned to σ and for all $t \geq s$, $\sigma \subseteq \Lambda^t$, σ is not initialized at stage t, $G^t_\sigma = G^s_\sigma$, $H^t_\sigma = H^s_\sigma$ and E^t_σ is M-coherent. Then there is a stage $r > s$ such that for all $t \geq r$, $\sigma^\frown\langle 0\rangle \subseteq \Lambda^t$, $\sigma^\frown\langle 0\rangle$ is not initialized at stage t, $G^t_{\sigma^\frown\langle 0\rangle} = G^r_{\sigma^\frown\langle 0\rangle}$, $H^t_{\sigma^\frown\langle 0\rangle} = H^r_{\sigma^\frown\langle 0\rangle}$ and $E^t_{\sigma^\frown\langle 0\rangle}$ is M-coherent. Furthermore, P_n is satisfied.*

Proof. As σ is not initialized at any stage $t \geq s$ and the only possible outcome for σ is $\sigma^\frown\langle 0\rangle$, the basic module for P_n will set $\sigma^\frown\langle 0\rangle \subseteq \Lambda^t$ for every $t \geq s$. Furthermore, as a node will not initialize itself, the only stage $t \geq s$ at which $\sigma^\frown\langle 0\rangle$ can be initialized is $t = s$. We note that Case 1 of the basic module for P_n will be followed at stage $t > s+1$ iff it is followed at stage $s+1$, and in this case, the lemma follows from the corresponding properties for σ. Otherwise, if s is chosen to be minimal, then Case 3 of the basic module for P_n will be followed at stage $s+1$, and Case 2 will be followed at every $t > s+1$. Thus we will set $G^t_{\sigma^\frown\langle 0\rangle} = G^{s+1}_{\sigma^\frown\langle 0\rangle}$ and will set $H^t_{\sigma^\frown\langle 0\rangle} \neq H^{t-1}_{\sigma^\frown\langle 0\rangle}$ at $t > s$ only when we revise the last character function to one which succeeds it in the $\preccurlyeq$ ordering. As this can happen only finitely often, there will be an $r > s$ after which this never happens. Now by Lemmas 4.16 and 4.10, $E^t_{\sigma^\frown\langle 0\rangle}$ will be M-coherent for all $t \geq s$ and P_n will be satisfied. □

We next pass to the satisfaction of Q_n.

Lemma 5.3. *Suppose that s and σ are given such that Q_n is assigned to σ and for all $t \geq s$, $\sigma \subseteq \Lambda^t$, σ is not initialized at stage t, $G^t_\sigma = G^s_\sigma$, $H^t_\sigma = H^s_\sigma$ and E^t_σ is M-coherent. Then there is a stage $r > s$ such that for all $t \geq r$, $\sigma^\frown\langle 0\rangle \subseteq \Lambda^t$, $\sigma^\frown\langle 0\rangle$ is not initialized at stage t, $G^t_{\sigma^\frown\langle 0\rangle} = G^r_{\sigma^\frown\langle 0\rangle}$, $H^t_{\sigma^\frown\langle 0\rangle} = H^r_{\sigma^\frown\langle 0\rangle}$ and $E^t_{\sigma^\frown\langle 0\rangle}$ is M-coherent. Furthermore, Q_n is satisfied.*

Proof. The proof is virtually identical to that of Lemma 5.2. We merely replace P_n with Q_n and Lemma 4.16 with Lemma 4.15. □

We now turn to the satisfaction of R_n.

Lemma 5.4. *Suppose that s and σ are given such that R_n is assigned to σ and for all $t \geq s$, $\sigma \subseteq \Lambda^t$, σ is not initialized at stage t, $G^t_\sigma = G^s_\sigma$, $H^t_\sigma = H^s_\sigma$ and E^t_σ is M-coherent. Then there is an $r > s$ such that for all $t \geq r$, $\sigma^\frown\langle 0\rangle \subseteq \Lambda^t$, $\sigma^\frown\langle 0\rangle$ is not initialized at stage t, $G^t_{\sigma^\frown\langle 0\rangle} = G^r_{\sigma^\frown\langle 0\rangle}$, $H^t_{\sigma^\frown\langle 0\rangle} = H^r_{\sigma^\frown\langle 0\rangle}$ and $E^t_{\sigma^\frown\langle 0\rangle}$ is M-coherent. Furthermore, R_n is satisfied.*

Proof. It follows from Definition 4.22(i) and (iii) that the speeding up instances of Step 2 of the basic module for R_n eventually terminate.

By hypothesis, we may fix a stage s_0 such that for all $t \geq s_0$, $\sigma \subset \Lambda^t$, σ is not initialized at stage t, $G^t_\sigma = G^s_\sigma$, $H^t_\sigma = H^s_\sigma$ and E^t_σ is M-coherent. For every $j < j(\sigma, s)$ let $I^{s_0}_{\sigma,k,j} = (p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1})$. For every j such that $[p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}]$ has property P, a number $n(\sigma, j, s) = n(p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}, E^s_\sigma, \widetilde{B}^s_{\sigma,j}, s)$ will be determined, and by Definition 4.22(i) and (iv), there will be a stage $s_1 \geq s_0$ such that for all $t \geq s_1$ and all $j < j(\sigma, s)$, the block and number designated by $\gamma(p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}, n(\sigma, j, s), t)$ will be the same as those designated by $\gamma(p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}, n(\sigma, j, s), s_1)$ and $\beta(p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}, t)$ will advise a search on $[p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}]$ at stage t. By Definition 4.22(ii), we can fix a j such that $[p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}]$ has property P.

First suppose that there is an $s_2 \geq s_1$ such that there is no current attack for σ at any stage $t \geq s_2$. By the choice of j and by Definition 4.22(ii), $(p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}) \cap M$ will contain at least two complete maximal blocks of M. The failure to start an attack on $I^s_{\sigma,k,j}$ can only be due to the fact that $|\{\widetilde{x} \in (\widetilde{p}^s_{\sigma,k,j}, \widetilde{p}^s_{\sigma,k,j+1}) \cap \widetilde{M} : \phi_n(\widetilde{x}) = 1\}| < 2$. But then ϕ_n is not a choice function for $\widetilde{M}$, so R_n is satisfied. Furthermore, we set $g^s_{k+1} = g^s_k$ and $h^s_{k+1} = h^s_k$ in this situation and follow outcome $\sigma^\frown\langle 01\rangle$, so by hypothesis, $E^s_{\sigma^\frown\langle 01\rangle}$ will be M-coherent, and will be the environment specified at all $t \geq s_2$. Furthermore, neither σ nor $\sigma^\frown\langle 0\rangle$ will be initialized at any $t \geq s_2$, so the lemma holds in this case.

Now suppose that there is a current attack at infinitely many stages $\geq s_1$. By choice of s_1, any such attack initiated at $t \geq s_1$ must include an attack on $I^t_{\sigma,k,j}$, and by choice of j, this attack is never canceled. Hence there is an $s_3 \geq s_1$ such that an attack on σ is initiated at some stage $\leq s_3$, and not canceled at any stage $t \geq s_3$. By Definition 4.22(iii) and (iv), $\gamma(p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}, n(\sigma,j,s), t)$ will produce the same block and number for this attack at all sufficiently large stages; and the same number will be produced and will determine the same h^t_{k+1} at all stages at which γ produces a number. By the construction, every extension $\widetilde{M}^t$ of $\widetilde{M}^{s_3}$ must respect h^t_{k+1} for the least $t \geq s_3$ at which $\gamma(p^s_{\sigma,k,j}, p^s_{\sigma,k,j+1}, n(\sigma,j,s), t)$ produces a block and number. But then by Lemmas 4.18, 4.20, and 4.10 $E^t_{\sigma^\frown\langle 00\rangle}$ must be M-coherent for each sufficiently large t, and no node in this basic module will act to initialize other nodes at any sufficiently large t. Furthermore, by (4.2)–(4.8), $\widetilde{B}^s_\sigma$ will be a block of $\widetilde{M}^t$ at all sufficiently large t, so as $|\{\widetilde{x} \in \widetilde{B}^s_\sigma : \phi_n(\widetilde{x}) = 1\}| \geq 2$, ϕ_n cannot be a choice function for $\widetilde{M}$. The lemma is now seen to hold in this case. □

As $g = \lim\{g_\sigma : \sigma \subset \Lambda\}$ and g_σ is always defined to be one-to-one and order-preserving, the theorem now follows by induction from Lemmas 5.2–5.4. □

The next two corollaries follow immediately from Theorem 5.1 and Lemmas 4.23 and 4.24 respectively.

Corollary 5.5. *Let α be completely infinite. Then α does not admit computable choice sets.*

Corollary 5.6. *Let $\alpha = \omega + \delta$ be a computable order-type such that for all $a \in \delta$, $(-\infty, a]$ has infinitely many blocks and δ has no least element. Then α does not admit computable choice sets.*

6. The Small Block Construction

There are two important differences between the Small and Large Block Constructions. The first has already been mentioned; the Large Block Construction builds a Δ^0_2 isomorphism between models and works with linear environments, while the Small Block Construction builds a Δ^0_3 isomorphism between models and works with ordinary environments. The other difference is in the strategy for satisfying the requirement R_n, which is more complex for the Small Block Construction. We cannot ensure that we will find a block of $\widetilde{M}$ for which ϕ_n selects two elements and which can be matched with a block of M. Thus if the strategy used for large blocks fails, we will use a large block to create a small maximal block of $\widetilde{M}$ for which ϕ_n fails to select any element, and match it with a maximal block of M.

The Small Block Property is not enough to ensure the success of the Small Block Construction. We will introduce an additional condition which is enough to ensure success in the case covering our main theorem. Afterwards, we will discuss modifications that work with different additional conditions, and how these can be used to obtain results in additional cases.

Requirements will be assigned to nodes of a tree of strategies, and at stage s of the construction, a node σ will be chosen and an attempt will be made to satisfy the requirement assigned to σ. The attempt will be made taking the *environment* E^s_σ of σ at stage s into account. We are now ready to state the Small Block Theorem.

Theorem 6.1. (Small Block Theorem) *Suppose that every component of every finite separation of α selected by P has a small maximal block. Then if there is a computable model of α which has SBP, then α does not admit computable choice sets.*

Proof. We first present the basic modules for satisfying requirements. The construction will be a Δ^0_3 construction. Nodes of higher priority than σ will provide us with an environment $E^{s-1}_\sigma = \langle M^{s-1}, \widetilde{M}^{s-1}, G^{s-1}_\sigma, H^{s-1}_\sigma\rangle$ for each stage s whenever σ is not initialized at the beginning of stage s, and it will be the case that $G^t_\sigma = G^r_\sigma$ and $H^t_\sigma = H^r_\sigma$ for all sufficiently large r and t. We assume, by induction, that all environments are coherent.

The basic modules for P_n and Q_n are identical to those for the Large Block Construction, so are not repeated here. We turn to the satisfaction of R_n. As $\mathcal{M}$ has SBP, we can fix the algorithms β and γ and q provided by SBP. The idea will be to first match a block $\widetilde{B}$ of $\widetilde{M}$ with a large block of M and hope that ϕ_n does not choose exactly one point of $\widetilde{B}$. If ϕ_n fails to cooperate, we revise $\widetilde{B}$ so that the first point of the revised block $\widetilde{D}$ is chosen by ϕ_n, and try again. If ϕ_n persists in choosing only one point from $\widetilde{D}$, a subblock of $\widetilde{D}$ which does not contain the chosen point can be matched with a small block of M. We will assume by induction that all environments are coherent and, at each stage s, there is at most one current attack at stage s in each interval I^s_j.

The basic module for $\mathbf{R_n}$: We work under the assumption that $\mathcal{M}$ has SBP. If σ is a node of our tree of strategies which is not initialized at the beginning of stage s and is a terminal node of a module, we let $\langle P^s_\sigma, \widetilde{P}^s_\sigma\rangle$ be the separation of $\langle M^s, \widetilde{M}^{s-1}\rangle$ induced by g^s_σ. We use $I^s_{\sigma,j} = [p^s_{\sigma,j}, p^s_{\sigma,j+1}] \cap M^s$ and $\widetilde{I}^s_{\sigma,j} = [\widetilde{p}^s_{\sigma,j}, \widetilde{p}^s_{\sigma,j+1}] \cap \widetilde{M}^{s-1}$ to represent the intervals determined by this separation.

The module will be infinite, but we will show that the true path has finite intersection with the module. We fix the initial node δ of the module. As we take extensions of nodes, we will only define an environment at a terminal node of the module. We fix a computable ω-ordering $\mathcal{O} = \langle \xi_\rangle \rangle$ of all finite sequences of finite sequences of natural numbers. The module will consist of stacks of submodules, each working with a particular element of $\mathcal{O}$; δ will work with the first element of $\mathcal{O}$.

We describe the submodule corresponding to a given element $\xi_i \in \mathcal{O}$ which begins at the node ρ. Given ρ, fix the lowest priority $\nu \supseteq \delta$ such that ν is a terminal node of a module at the end of stage $s-1$ and either $\nu = \rho$ or ν has higher priority than ρ. ($\nu = \rho$ only when $\rho = \delta$, and δ will always be a terminal node of a module.) Let $j(\nu, s)$ be the length of the separation at ν at stage s. There are several steps at stage s.

Step 1: (Determine Attack Intervals and Parameters) This step takes place at ρ, which has two outcomes $0 <_L 1$. 0 will indicate that this a current attack, and 1 will indicate that we have decided not to begin attacks.

Substep 1.1: (Determine the Parameters m, n and U) Let $U^s_{\rho,j} = \text{dom}(g^s_{\nu,j})$ for all $j < j(\nu, s)$. Two consecutive elements $c < d \in U^s_{\rho,j}$ are *constrained to be in the same block* at stage s iff they are not certifiably separated in M^s. The set of blocks $\widehat{\mathcal{B}}^s_{\rho,j}$ consists of all minimal intervals of $I^s_{\nu,j}$ whose endpoints are in $U^s_{\rho,j}$ and such that if a block contains consecutive elements of $U^s_{\rho,j}$, then those elements are constrained to be in the same block at stage s. We set $m^s_{\rho,j} = |\widehat{\mathcal{B}}^s_{\rho,j}| - 1$. (The number $m^s_{\rho,j}$ will be the number of blocks which require matching. The first and last blocks of $\widehat{\mathcal{B}}^s_{\rho,j}$ generate blocks of $\widetilde{I}^s_{\nu,j}$ which will not require matching, and there will be an additional block for which matching will be required.) We set $n^s_{\rho,j} = q(0)$ if $q(n) < n$ for some n, and we set $n^s_{\rho,j} = 1 + \max\{|B| : B \in \widehat{\mathcal{B}}^s_{\rho,j}\}$ otherwise. If $q(n) < n$ for some n, then we speed up the enumeration of M, if necessary, until we find a stage at which all blocks in $\widehat{\mathcal{B}}^s_{\rho,j}$ are $\leq q(0)$-blocks. Now go to Substep 1.2.

Substep 1.2: (Determine the Target Intervals) The target intervals are intervals of the separation of the environment for δ, rather than that for ν. We fix the largest stage $t_0 < s$ at which all previous attacks for this submodule were canceled, and set $t_0 = 0$ if there was no such stage. An interval $I^s_{\delta,j}$ is a *target interval* at stage s if $\beta(p^s_{\nu,j}, p^s_{\nu,j+1}, m^s_{\rho,j}, n^s_{\rho,j}, t)$ declares that a search should take place on $I^s_{\delta,j}$ for all t such that $t_0 < t \leq s$. We speed up the enumeration of M, if necessary, until we find at least one target interval; we assume that the stage at which this interval is found is s. Now go to Substep 1.3.

Substep 1.3: (Determine the Small Blocks) Let $\xi_{i(\rho)}$ be the element of $\mathcal{O}$ currently selected for ρ. We select the first element of $\mathcal{O}$ if $\rho = \delta$; otherwise, the element is determined at an earlier stage. If $\xi_{i(\rho)}$ does not have exactly one sequence $B^s_{\rho,j}$ from each target interval $I^s_{\delta,j}$, or if $B^s_{\rho,j}$ does not have cardinality $< n^s_{\rho,j}$ or is not certifiably separated from every point in $U^s_{\delta,j}$, then we follow outcome $\rho^\frown\langle 1\rangle$ which is a terminal node of this submodule. We begin a new submodule at $\rho^\frown\langle 1\rangle$ *selecting* the next element of $\mathcal{O}$, but do not define an environment for $\rho^\frown\langle 1\rangle$ at stage s. Otherwise, we follow outcome $\sigma = \rho^\frown\langle 0\rangle$ and go to Step 2.

Step 2: (Determine the Maximality of the Selected Blocks) This step takes place at σ, which has two outcomes $0 <_L 1$. There will be a stage $t(\sigma, s) \leq s$ which is associated with σ and is periodically reset. $t(\sigma, s) = 0$ if either $s = 0$ or σ is initialized at the end of stage $s - 1$. If, for some target interval $I^s_{\delta,j}$, there is a block C_j of $M^{t(\sigma,s-1)}$ properly containing $B^s_{\rho,j}$ such that C_j is also a block of M^s (so we have not succeeded in finding new elements of M which separate the elements in $B^s_{\rho,j}$ from those in C_j), then we follow outcome $\sigma^\frown\langle 1\rangle$ which is a terminal node of this submodule. We set $t(\sigma, s) = t(\sigma, s - 1)$ and begin a new submodule at $\sigma^\frown\langle 1\rangle$ working with the next element of $\mathcal{O}$, but do not define an environment for $\rho^\frown\langle 1\rangle$ at stage s. Otherwise, we reset $t(\sigma, s) = s$, follow outcome $\tau = \sigma^\frown\langle 0\rangle$ and go to Step 3.

Step 3: (Determine Search Parameters) We begin this step at node τ which has two outcomes $0 <_L 1$. There are several substeps.

Substep 3.1 (Determine Set of Blocks Requiring Storage) We set $\widetilde{\mathcal{B}}^s_{\tau,j}$ to be the set of all subblocks of $\widetilde{I}^s_{\nu,j}$ of the form $[g^s_\nu(c), g^s_\nu(d)] \cap \widetilde{M}^{s-1}$ such that $c < d$ are the endpoints of a block in $\widehat{\mathcal{B}}^s_{\rho,j}$ and neither c nor d is an endpoint of $I^s_{\nu,j}$. These are the blocks of $\widetilde{I}^s_{\nu,j}$ *requiring storage*. List the blocks of $\widetilde{\mathcal{B}}^s_{\tau,j}$, under the ordering of $\widetilde{M}^{s-1}$, as $\langle \widetilde{B}^s_{\tau,j,v}\rangle$. We now go to Substep 3.2.

Substep 3.2 (Search for Storage) We now speed up the enumeration of M until every target interval $I^s_{\delta,j}$ encounters one of the following two situations:

(i) $\beta(p^s_{\nu,j}, p^s_{\nu,j+1}, m^s_{\rho,j}, n^s_{\rho,j}, s)$ advises that $I^s_{\delta,j}$ should no longer be a target interval.

(ii) $\gamma(p^s_{\nu,j}, p^s_{\nu,j+1}, U^s_{\rho,j}, m^s_{\rho,j}, n^s_{\rho,j}, s)$ successfully completes the search.

We *cancel* the attack on $I^s_{\delta,j}$ if (i) holds for $I^s_{\delta,j}$. If the attacks are canceled for all target intervals $I^s_{\delta,j}$, then we conclude this stage and set $\Lambda^s = \rho$. Otherwise, whenever (i) holds for $I^s_{\delta,j}$, then $I^s_{\delta,j}$ *loses its status as a target interval*. Denote the set of intervals produced by $\gamma(p^s_{\nu,j}, p^s_{\nu,j+1}, U^s_{\rho,j}, m^s_{\rho,j}, n^s_{\rho,j}, s)$ as $\mathcal{B}^s_{\tau,j} = \langle B^s_{\tau,j,v}\rangle$. Now go to Substep 3.3.

Substep 3.3: (Initiate Attacks and Define New Environments) We *initiate an attack* for each target interval $I^s_{\delta,j}$ at stage s for which there is no current attack. For each target interval $I^s_{\delta,j}$ for which there is now a current attack, we define an environment $\widehat{E}^s_{\tau,j}$. If there was an attack on $I^s_{\delta,j}$ at stage $s-1$ which has not been canceled, then an environment $\widehat{E}^{s-1}_{\tau,j} = \langle M^{s-1}, \widetilde{M}^{s-1}, G^{s-1}_{\tau,j}, H^{s-1}_{\tau,j}\rangle$ will have been defined and will be coherent. By Lemma 4.10, we can revise $H^{s-1}_{\tau,j}$ to $H^s_{\tau,j}$ so that $\widehat{E}^s_{\tau,j}$ defined as $\langle M^s, \widetilde{M}^{s-1}, G^{s-1}_{\tau,j}, H^s_{\tau,j}\rangle$ will be coherent. Otherwise, we define an environment $\widetilde{E}^s_{\tau,j}$ as follows. Let $E^s_\nu = \langle M^s, \widetilde{M}^{s-1}, G^s_\nu, H^s_\nu\rangle$, with $G^s_\nu = \langle g^s_{\nu,r} : r \leq k\rangle$ and $H^s_\nu = \langle h^s_{\nu,r} : r \leq k\rangle$, and fix $r_0 \leq k$ such that $G^s_\delta = \langle g^s_{\nu,r} : r \leq r_0\rangle$. Recall that $\mathcal{B}^s_{\tau,j} = \langle B^s_{\tau,j,v}\rangle$.

We now try to locate the block $B^s_{\rho,j}$, under the ordering of M, between consecutive elements of $\mathcal{B}^s_{\tau,j}$. Fix the smallest v_0 such that all elements of $B^s_{\rho,j}$ are less that the smallest element of B^s_{τ,j,v_0}. Define g^s_{τ,j,r_0+1} to be the extension of g^s_{ν,r_0} obtained by mapping the elements of an initial segment of $B^s_{\tau,j,v}$ to those of $\widetilde{B}^s_{\tau,j,v+1}$ in a one-to-one order-preserving way for all $v > v_0$, mapping the elements of an initial segment of $B^s_{\tau,j,v}$ to those of $\widetilde{B}^s_{\tau,j,v}$ in a one-to-one order-preserving way for all $v < v_0$, and inserting a new block $\widetilde{D}^s_{\tau,j}$ of consecutive elements into $\widetilde{M}^s$ which are the images of those in B^s_{τ,j,v_0}. In addition, we map $e \in U^s_{\rho,j}$ which lies in the first or last block of $\widehat{\mathcal{B}}^s_{\rho,j}$ to $g^s_\nu(e)$, and insert new elements into $\widetilde{M}^s$, if necessary to be images of the remaining elements of $U^s_{\rho,j}$ so that the resulting map is one-to-one and order-preserving. We now speed up the enumeration of M until (4.1) again holds. If $I^s_{\delta,j}$ is not a target interval, we set $g^s_{\tau,j,r_0+1} = g^s_{\nu,k}$ and let the character $h^s_{\tau,j,r_0+1} = h^s_{\nu,k}$. If $I^s_{\delta,j}$ is a target interval, define the values $h^s_{\tau,j,r_0+1}(w)$ of the character h^s_{τ,j,r_0+1} for the wth interval whose endpoints $\widetilde{c} = g^s_{\tau,j,r_0+1}(c), \widetilde{d} = g^s_{\tau,j,r_0+1}(d)$ are in $\mathrm{rng}(g^s_{\tau,j,r_0+1})$ as follows: if c and d are certifiably separated in M^s, define $h^s_{\tau,j,r_0+1}(w) = 2$; if $\widetilde{c}$ and $\widetilde{d}$ both lie in the same block of $\widetilde{\mathcal{B}}^s_{\tau,j}$ and this block is not the first or last block of $\widetilde{\mathcal{B}}^s_{\tau,j}$, then define $h^s_{\tau,j,r_0+1}(w) = 1$; if $\widetilde{c}$ and $\widetilde{d}$ both lie in the same block of $\widetilde{\mathcal{B}}^s_{\tau,j}$ and this block is the first or last block of $\widetilde{\mathcal{B}}^s_{\tau,j}$, and $h^s_{\nu,k}$ assigns the value 1 to the block $[\widetilde{c}, \widetilde{d}] \cap \widetilde{M}^{s-1}$, then define $h^s_{\tau,j,r_0+1}(w) = 1$; otherwise, we set $h^s_{\tau,j,r_0+1}(w) = 0$. g^s_{τ,j,r_0+1} and h^s_{τ,j,r_0+1} agree with $g^s_{\nu,k}$ and $h^s_{\nu,k}$, respectively, except on $I^s_{\delta,j}$.

We let $\widehat{G}^s_{\tau,j} = \langle g^s_{\nu,r} : r \leq r_0\rangle ^\frown \langle g^s_{\tau,j,r_0+1}\rangle$ and $\widehat{H}^s_{\tau,j} = \langle h^s_{\nu,r} : r \leq r_0\rangle ^\frown \langle h^s_{\tau,j,r_0+1}\rangle$ and $\widehat{E}^s_{\tau,j} = \langle M^s, \widetilde{M}^s, \widehat{G}^s_{\tau,j}, \widehat{H}^s_{\tau,j}\rangle$. We now let $\widehat{E}^s_\tau$ be the piecewise extension of E^s_ν via $\langle \widehat{E}^s_{\tau,j}\rangle$, and go to Substep 3.4.

Substep 3.4 (Wait for Convergence) If we have previously passed to Step 4 for the current attack, we do so immediately. Otherwise, we ask if $\phi^s_n(\widetilde{x}) \downarrow \in \{0,1\}$ for all $\widetilde{x} \in \widetilde{D}^s_{\tau,j}$ and $|\{x \in \widetilde{D}^s_{\tau,j} : \phi^s_n(\widetilde{x}) = 1| = 1$. If we receive an affirmative answer for each target interval $I^s_{\delta,j}$, we proceed to Step 4, following outcome $\eta = \tau ^\frown \langle 0\rangle$. Otherwise, we follow outcome $\tau ^\frown \langle 1\rangle$ which becomes a terminal node of the module, and set $E^s_{\tau ^\frown \langle 1\rangle} = \widehat{E}^s_\tau$.

Step 4: (Normalize) We begin this step at node η which has two outcomes $0 <_L 1$. (We wish to now force the creation of a small block on which ϕ_n makes no selection; we do so by normalizing $\widetilde{D}^s_{\tau,j}$ by requiring that $\phi^s_n(x) = 1$ for the smallest x of this block. The remainder of the block will then be matched with $B^s_{\rho,i}$, if needed to satisfy R_n.) Place new elements into $\widetilde{M}^s$, if needed, right after the last element of $\widetilde{D}^s_{\tau,j}$, so that the block $\widehat{D}^s_{\tau,j}$ of $\widetilde{M}^s$ beginning with the unique element x of $\widetilde{D}^s_{\tau,j}$ such that $\phi^s_n(x) = 1$ has x as its first element, and has only elements of $\widetilde{D}^s_{\tau,j}$ and the elements just placed into $\widetilde{M}^s$. We now proceed as in Substep 3.4 for η in place of τ, replacing $\widetilde{D}^s_{\tau,j}$ with $\widehat{D}^s_{\tau,j}$ and revising $\widehat{E}^s_{\tau,j}$ accordingly, and acting as follows upon receiving an affirmative answer. Let $\widetilde{B}^s_{\tau,j}$ be the subblock of $\widetilde{D}^s_{\tau,j}$ beginning with the second element of the latter block, and of cardinality $|B^s_{\tau,j}|$. The existence of this block follows from Substep 3.2(i). We follow outcome $\eta ^\frown \langle 0\rangle$ which becomes a terminal node of the module. The environment $E^s_{\eta ^\frown \langle 0\rangle}$ is the modification of E^s_τ obtained by proceeding as in Substep 3.4, except that we match $B^s_{\tau,j}$ with $\widehat{B}^s_{\tau,j}$ instead of matching B^s_{τ,j,v_0} with $\widetilde{D}^s_{\tau,j}$.

We now define our tree of strategies in the usual way by specifying a computable ordering of all requirements, and assign requirements to the trees, in order, through stacks of basic modules. If some current environment E^{s-1}_σ is not coherent at stage s, we pick the highest priority such σ and jump directly to the beginning of the basic module (or submodule) which defined that environment. Otherwise, the construction at stage s follows the description of the basic modules, proceeding from node to node, initializing nodes to the right of the current path, and stopping and specifying the *current path* when it specifies an outcome for a node which is in the initialized state. All uninitialized nodes to the left of the current path set all their parameters at stage s identically to the way they were set at the end of stage $s-1$. Let Λ^s be the current path at stage s, and let Λ be the *true path* for the construction, i.e. the highest priority path, each of whose nodes is contained in Λ^s for infinitely many s. Λ^s is *uninitialized* at stage $s+1$.

We must now prove that $|\Lambda| = \infty$ and that all requirements are satisfied. The proof will use the next three lemmas, one for each type of requirement.

We begin with the satisfaction of P_n.

Lemma 6.2. *Suppose that s and σ are given such that P_n is assigned to σ, $\sigma \subseteq \Lambda^t$ for infinitely many t, and for all $t \geq s$, σ is not initialized at stage t, $G^t_\sigma = G^s_\sigma$, $H^t_\sigma = H^s_\sigma$ and E^t_σ is M-coherent. Then there is an $r > s$ such that for all $t \geq r$, $\sigma^\frown\langle 0\rangle \subseteq \Lambda^t$, $\sigma^\frown\langle 0\rangle$ is not initialized at stage t, $G^t_{\sigma^\frown\langle 0\rangle} = G^r_{\sigma^\frown\langle 0\rangle}$, $H^t_{\sigma^\frown\langle 0\rangle} = H^r_{\sigma^\frown\langle 0\rangle}$ and $E^t_{\sigma^\frown\langle 0\rangle}$ is M-coherent. Furthermore, P_n is satisfied.*

Proof. Identical to the proof of Lemma 5.2. □

We next pass to the satisfaction of Q_n.

Lemma 6.3. *Suppose that s and σ are given such that Q_n is assigned to σ and $\sigma \subseteq \Lambda^t$ for infinitely many t, and for all $t \geq s$, σ is not initialized at stage t, $G^t_\sigma = G^s_\sigma$ and, $H^t_\sigma = H^s_\sigma$ and E^t_σ is M-coherent. Then there is an $r > s$ such that for all $t \geq r$, $\sigma^\frown\langle 0\rangle \subseteq \Lambda^t$, $\sigma^\frown\langle 0\rangle$ is not initialized at stage t, $G^t_{\sigma^\frown\langle 0\rangle} = G^r_{\sigma^\frown\langle 0\rangle}$, $H^t_{\sigma^\frown\langle 0\rangle} = H^r_{\sigma^\frown\langle 0\rangle}$ and $E^t_{\sigma^\frown\langle 0\rangle}$ is M-coherent. Furthermore, Q_n is satisfied.*

Proof. Identical to the proof of Lemma 5.3. □

We now turn to the proof of the satisfaction of R_n. During that proof, we will frequently apply a lemma which implies that an environment $\langle M^s, \widetilde{M}^{s-1}, G, H\rangle$ is coherent, and instead claim that $E = \langle M^s, \widetilde{M}^s, G, H\rangle$ is coherent. This will be done when the elements of $\widetilde{M}^s - \widetilde{M}^{s-1}$ are added to obtain a coherent environment extending E, so E will also be coherent by definition.

Lemma 6.4. *Suppose that s and σ are given such that R_n is assigned to σ, $\sigma \subseteq \Lambda^t$ for infinitely many t, and for all $t \geq s$, σ is not initialized at stage t, $G^t_\sigma = G^s_\sigma$, $H^t_\sigma = H^s_\sigma$ and E^t_σ is M-coherent. Then there is an $r > s$ such that for all $t \geq r$, $\sigma^\frown\langle 0\rangle \subseteq \Lambda^t$, $\sigma^\frown\langle 0\rangle$ is not initialized at stage t, $G^t_{\sigma^\frown\langle 0\rangle} = G^r_{\sigma^\frown\langle 0\rangle}$, $H^t_{\sigma^\frown\langle 0\rangle} = H^r_{\sigma^\frown\langle 0\rangle}$ and $E^t_{\sigma^\frown\langle 0\rangle}$ is M-coherent. Furthermore, R_n is satisfied.*

Proof. We show that Λ has finite intersection with the basic module for R_n, that R_n is satisfied, and that the submodule used to satisfy R_n yields a coherent sequence. We analyze the steps of the basic module for R_n, beginning with an analysis of a fixed submodule beginning at $\rho \subset \Lambda$.

We first note that each time Λ^s jumps to a node λ extending a terminal node κ of the submodule beginning at ρ because the environment defined for λ is not coherent, it initializes λ. Thus this can happen only finitely often for κ unless the construction passes through κ infinitely often without jumping to such a λ. Hence we may assume that if such an outcome κ is contained in Λ, then κ is initialized only finitely often in this way.

A particular attack will remain in force until canceled. By our hypotheses on α and Definition 4.25(i),(iii), we will have a stage s such that no attack begun on an interval which has property P is canceled by β after stage s; and all attacks on intervals failing to have property P are either eventually canceled or are successfully completed with γ providing a suitable sequence of blocks. As the set of intervals for which we perform searches is fixed by δ, we can assume that we have reached a stage after which β does not change the status of intervals having property P as target intervals for this attack. If the sequence from $\mathcal{O}$ does not select a sufficiently small block from some target interval, or the elements of such a block are not certifiably separated from the designated set U, we will know this at all sufficiently large stages, follow outcome $\rho^\frown\langle 1\rangle$ at all sufficiently large stages, and go to the next submodule, completing our proof for this submodule. Note that by Definition 4.25(v), the speed up when $q(n) < n$ for some n will eventually terminate successfully. Hence we may assume that the sets $B^s_{\rho,j}$ selected are blocks of the target intervals $\widetilde{I}^s_{\delta,j}$. If a block is selected from a non-target interval and fails to have the desired properties, then β will decertify that interval as a target interval at some stage after the last attacks were all canceled, so we will be able to pass to Step 2.

If a block $B^s_{\rho,j}$ is not maximal, then we will recognize this situation at all sufficiently large stages and follow outcome $\sigma^\frown\langle 1\rangle$, passing to the next submodule. Thus we can assume that we pass to node τ at Step 3 infinitely often.

We have already shown that there is a stage after which κ is never initialized, and note that τ will never not be initialized thereafter. Hence the set of blocks determined in Substep 3.1 and the storage needs determined in Substep 3.2 will not change as we pass from stage to stage. By hypothesis, β will never again change the status of intervals having property P, so we will eventually have an attack which β never cancels, so will not set $\Lambda^s = \rho$ at Substep 3.2. By Definition 4.25(iv), γ will produce sequences of blocks for all attacks begun at sufficiently large stages on intervals having property P, so we will eventually go on to Substep 3.3 for the non-canceled attacks. As the blocks produced by γ are always pairwise certifiably separated and also certifiably separated from the points in the corresponding set U, the intervals between these blocks and immediately before the first block and immediately following the last block will be infinite, so the speed up in Substep 3.3 will eventually produce a stage at which (4.1) again holds, and we will pass to Substep 3.4.

Suppose that we follow outcome $\tau^\frown\langle 1\rangle$ at Substep 3.4 which is a terminal node for the module. We note that either $\nu = \delta$, or ν extends a terminal node κ of the basic module which begins at δ in which case $E^s_\nu \supseteq E^s_\kappa$. Hence E^s_ν respects all elements of $\widetilde{\mathcal{B}}^s_{\rho,j}$ as well as $\widetilde{D}^s_{\tau_j}$ for all j. By Lemmas 4.21 and 4.10, $E^s_{\tau^\frown\langle 1\rangle}$ will be an M-coherent environment for all sufficiently large s. Now ϕ_n cannot be a choice function as it is either not total or selects more than one element from some block of $\widetilde{M}$, so R_n will be satisfied, and we will have reached a terminal node of the module. Hence we now need to consider the passage to Step 4.

Suppose that we follow outcome $\eta^\frown\langle 1\rangle$ at Step 4 which is a terminal node for the module. We note that either $\nu = \delta$, or ν extends a terminal node κ of the basic module which begins at δ in which case $E^s_\nu \supseteq E^s_\kappa$. Hence E^s_ν respects all elements of $\widetilde{\mathcal{B}}^s_{\rho,j}$ as well as $\widehat{D}^s_{\tau_j}$ for all j. By Lemmas 4.21 and 4.10, $E^s_{\tau^\frown\langle 1\rangle}$ will be an M-coherent environment for all sufficiently large s. In this case, ϕ_n cannot be a choice function as it is either not total or selects more than one element from some block of $\widetilde{M}$, so R_n will be satisfied.

Now suppose that we follow outcome $\eta^\frown\langle 0\rangle$ at Step 4, which is a terminal node for the module. We note that either $\nu = \delta$, or ν extends a terminal node κ of the basic module which begins at δ in which case $E^s_\nu \supseteq E^s_\kappa$. Hence E^s_ν respects all elements of $\widetilde{\mathcal{B}}^s_{\rho,j}$ as well as $\widetilde{B}^s_{\tau_j}$ for all j. By Lemmas 4.21 and 4.10, $E^s_{\tau^\frown\langle 0\rangle}$ will be an M-coherent environment for all sufficiently large s. In this case, ϕ_n cannot be a choice function as it does not select any elements from some maximal block of $\widetilde{M}$.

By our additional hypothesis on α that every component of every finite separation contains a small maximal block, and as δ is identical for all submodules, it follows from Definition 4.25(ii) that there is an element of $\mathcal{O}$ for which we will never follow outcome $\rho^\frown\langle 1\rangle$ or $\sigma^\frown\langle 1\rangle$. The lemma now follows from the above discussion. $\square$

We note that for all $\sigma \subset \Lambda$, if $s(\sigma)$ is the smallest stage s such that $E^t_\sigma = E^s_\sigma$ for all $t \geq s$, then if a new environment E^t_τ is specified at some stage $t \geq s(\sigma)$ where $\tau \subset \Lambda$ is a terminal node of the module beginning at σ, then E^t_τ is coherent, so $\widetilde{M}^t$ respects all intervals of the separation of $\widetilde{M}$ induced by g^t_τ of character 1. Hence by repeated applications of Lemma 4.10 and as any increasing sequence under $\succcurlyeq$ is finite, E^t_σ must be M-coherent for all sufficiently large t. As $g = \lim\{g_\sigma : \sigma \subset \Lambda\}$ and g_σ is always defined to be one-to-one and order-preserving, the theorem now follows by induction from Lemmas 6.2–6.4. □

The next two corollaries follow immediately from Theorem 6.1 and Lemmas 4.26 and 4.27 respectively.

Corollary 6.5. *Assume that α is n-bounded but not finite-constant and that every non-finite-constant component of every finite separation of α is n-bounded but not $n-1$-bounded and has a maximal $< n$-block. Then α does not admit computable choice sets.*

Corollary 6.6. *Let α be hereditarily unbounded with no completely infinite closed subinterval, and assume that α is not finite-constant and for all n, α has no n-bounded closed interval which is not m-constant for some m. Then α does not admit computable choice sets.*

We are now ready to prove the partial converse to Theorem 2.5.

Theorem 6.7. *Suppose that α is a non-finite-bounded computable order-type and that $1 + \alpha + 1$ is not finite-constant. Then there is a model of α which does not have a computable choice set.*

Proof. We proceed by cases.

Case 1: $1 + \alpha + 1$ has an interval $1 + \alpha_0 + 1$ which is completely infinite. The theorem follows from Corollary 5.5 and Lemma 2.2 in this case.

Case 2: $1 + \alpha + 1$ has an interval of the form $\omega + \delta$ as in Corollary 5.6. The theorem follows from Corollary 5.6 and Lemma 2.2 in this case.

Case 3: $1 + \alpha + 1$ has an interval of the form $\delta^* + \omega^*$, where $\omega + \delta$ is as in Corollary 5.6. The theorem follows from Corollary 5.6, Lemma 2.2, and Lemma 2.1 in this case.

Case 4: Otherwise. As $1 + \alpha + 1$ is not finite-bounded and as Case 1 does not apply, in every finite separation of α, every component which has infinitely many maximal blocks must have infinitely many finite maximal blocks. Now if every finite separation of α has a component which has infinitely many infinite maximal blocks, then the theorem follows from Corollary 6.6 and Lemma 2.2.

Otherwise, α must have only finitely many infinite maximal blocks. But then, without loss of generality, we may assume that all endpoints of such blocks are in the separation, and that the separation includes one point from each infinite block. By the inclusion of endpoints of the infinite blocks if they exist, we must either have a closed subinterval of α satisfying the conditions of Case 2 or Case 3, or α has no infinite blocks. Hence we can assume that α has no infinite blocks. Now as α is not m-bounded for any m, the theorem follows from Corollary 6.6 and Lemma 2.2 in this case. □

Our main result now follows.

Theorem 6.8. *A non-finite-bounded computable order-type α admits computable choice sets iff α is finite-constant.*

Proof. Immediate from Theorem 2.5, Theorem 6.7, and Corollary 2.3. □

While we have not been able to obtain a complete characterization in the finite-bounded case, some modifications to Theorem 6.1 provide many cases for which no computable choice set exists. For the discussion below, α will be a computable n-bounded order-type which has no finite separation for which each component is either constant, or has only finitely many maximal blocks, or is $(n-1)$-bounded. The first case follows immediately from Corollary 6.5.

Example 6.9. Suppose that α has no subinterval of order-type $n \cdot \eta$. Then α does not admit computable choice sets.

We have shown that all computable finite-bounded α have SBP. But the proof of Theorem 6.1 also made use of the hypothesis that every component interval of every finite separation of α has a small maximal block, so SBC alone is not sufficient. We note, however, that the construction can be modified to handle other cases.

The main conflict that needs to be resolved in the construction is the interplay between attempts at finding sufficient storage and identifying when a potential win for requirement R_n has actually selected a small block. β allows us to identify intervals with sufficient storage in a Σ_2 manner, but the corresponding submodule may or may not have identified a small block. Small blocks look small infinitely often, but they may also seem to be subblocks of large blocks infinitely often. If we blindly search for storage on an interval for which storage does not exist, the construction will terminate before satisfying all requirements, as the search will run forever. If we allow our construction to act on the first interval for which we find storage, then this interval may not have identified a small block yet (in fact, there may be none in the interval), and in order to preserve the isomorphism in case the block is, in fact, small, we will need to increase the storage requirements for the next submodule. Hence by the time we locate a small block on the correct interval, the increased storage needs may cause us to first find storage on a different interval. Thus we may make infinitely many attempts to satisfy a given requirement, and each supposedly successful attempt may be for a small block which is part of a large block. We succeeded above by arguing that the conditions on α provide us with an attempt for which the guesses on all intervals are true small maximal blocks, so the interval on which we find storage for that attempt produces a maximal block for which the requirement is satisfied.

The conflict discussed can be resolved in other ways. For example, if β is independent of m and its Σ_2 outcome selects only intervals with infinite storage capacity, then we can make attempts using β, one potential small block at a time. When β stabilizes, the storage requirements will cease to increase so we can carry out a search until storage is found on an interval selected by β. Step 2, where we determine whether a block is, in fact maximal, is not necessary, as we need only determine whether we have a block in this context. A revision of the construction in this way enables us to handle the following cases.

Example 6.10. If α has a subinterval of order-type $k \cdot \eta + m \cdot \eta$ where $k \neq m$, then α does not admit computable choice sets. For $k \cdot \eta + 1 + m \cdot \eta$ is also a computable order-type having a computable model with a computable choice set. Let β always select only the interval containing the element of the model corresponding to the interpolated point. Then this interval has the desired properties.

Definition 6.11. We say that α is *uniform* if α has order-type $k \cdot \eta$ for some k. α is *not uniformly dense* if α has no greatest element and no uniform final interval. A maximal block of a model of α is called a *separator* if it does not lie in the interior of a uniform interval.

Example 6.12. If $\alpha = \delta + 1$ and δ is not uniformly dense, then α does not admit computable choice sets. For if β picks just the last interval of α, then that interval has all the desired properties.

Example 6.13. Suppose that there is a number r such that in each finite separation of α, there are $\leq r$-many n-blocks which are separators (i.e., do not lie within a subinterval of order-type $n \cdot \eta$). Then α does not admit computable choice sets. In this case, we let β select all intervals with more than r distinct n-blocks, and note that any such interval provides infinite storage.

Note that the set of all examples which do not admit computable choice sets is closed under the * operation, by Lemma 2.1. Note further that if an example can be embedded as a closed interval in a computable model of a new order-type β, then β will also fail to admit computable choice sets.

The preceding examples and remarks allow us to cover all cases except for the following two cases: $\alpha = \delta + k \cdot \eta$ and $\alpha = \delta + \gamma$ where δ and γ^* are not densely uniform. However, we can further modify our construction so that it applies to some of these order-types.

The requirement that the intervals selected by β be identified in a Σ_2 manner can be weakened somewhat; we need only that the formula determining the selected intervals be the conjunction of Σ_2 and Π_2 formulas. We begin to search for storage when the first formula seems to have a Σ_2 outcome, beginning a new search each time the guess at the Σ_2 outcome is seen to be false, with increased storage requirements. After the first formula has found its Σ_2 outcome, the storage requirements will only be allowed to increase when the second formula seems to have a Σ_2 outcome. Each time a guess at the Σ_2 outcome of the second formula is seen to be false, these secondary increases of storage requirements are canceled. This will ensure that storage requirements on the intervals selected by β increase only finitely often. A revision of the construction in this way enables us to handle the following cases.

Example 6.14. Suppose that $\alpha = \delta + k \cdot \eta$ or $\alpha = \delta + \gamma$ where δ and γ^* are not densely uniform and δ is $(n-1)$-bounded. Then α does not admit computable choice sets. For if let β select only the leftmost component interval containing an n-block and define *small* to be of cardinality $< n$, then β will have the desired properties.

Example 6.15. Suppose that $\alpha = \delta + k \cdot \eta$ where $k < n$ and δ is not uniformly dense. Then α does not admit computable choice sets. For if let β select only the rightmost component interval containing an n-block and define *small* to be of cardinality $< n$, then β will have the desired properties.

The remaining cases can be described as follows. Every open cut in α which is not within a uniform interval separates α into $\gamma + \delta$, and every interval of α

containing elements to the left and right of the cut has n-blocks to the left and right of the cut. If, for one such cut, either γ or δ is uniform, it seems plausible to us to be able construct a computable model of $\gamma + 1 + \delta$ and so use one of the above strategies. But we have no idea of how to try to approach those α for which both γ and δ^* are not uniformly dense.

There is one more case, requiring more drastic modifications to the construction of Theorem 6.1, that we wish to discuss. This is the case in which there are only two sizes of maximal blocks. The success of the construction of Theorem 6.1 depends heavily on finding an interval with a small block which has sufficient storage. The role of β is to provide such an interval. The cases covered so far assume that either β is independent of m and so picks out intervals with infinite storage, or as in the cases to which we directly applied the theorem, that as the demand for the number of storage blocks increases, the intervals which provide only finitely many storage blocks are never chosen for any sufficiently large m, and also that we work on all such intervals to ensure success on at least one. When there are only two sizes of maximal blocks, the search for storage on an interval requires only one new large block, and its failure tells us that the only new blocks in that interval which require storage are small blocks; and we can always find storage for arbitrarily many small blocks. Thus:

Example 6.16. Suppose that there are only two cardinalities of maximal blocks of α. Then α does not admit computable choice sets.

The same idea can be used to handle the case in which α has infinitely many small blocks which are separators.

Julia Knight suggested a modification of problems of this sort, due to McCoy, in which complete characterizations seem to be more readily obtained. The idea is to look at all models of order-type α (α is computable) and require that the object being sought be computable from the degree of the model of order-type α. In the setting of this paper, we can ask the following question: For which computable order-types α is it true that every model B of order-type α has a choice set of degree less than or equal to the degree of B? This question has the expected answer.

Theorem 6.17. *Let α be a computable order-type. Then every model B of order-type α has a choice set of degree less than or equal to the degree of B iff α is finite-constant.*

Proof. We note that the proof of Theorem 2.5 relativizes to show that if α is finite-constant, then every model B of order-type α has a choice set of degree less than or equal to the degree of B.

For the converse direction, we note that all of our theorems relativize, by virtually identical proofs using oracles, to any degree; the starting model M is computable, but the model $\widetilde{M}$ and the enumerations are computable in the given degree. In fact, by the unrelativized Corollary 6.5, it follows that if every model B of order-type α has a choice set of degree less than or equal to the degree of B then α must be finite-constant.

We next note that Theorem 6.1 relativizes, and that the construction relativized to an oracle of degree $\mathbf{d}$ will yield a model of degree $\leq \mathbf{d}$ which has no choice function of degree $\leq \mathbf{d}$ as long as, given any finite separation of M, we can uniformly and effectively from $\mathbf{d}$ identify intervals which are not finite-constant and have infinitely many small blocks and infinitely many large blocks. We proceed by cases, indicating how to find such an interval in each case. Withoul loss of generality, we assume that α has both left and right endpoints.

Case 1: α has a subinterval of ordertype $\delta + 1$ or $(\delta + 1)^*$, where δ is not uniformly dense. By Example 6.12 and Lemmas 2.1 and 2.2, α does not admit computable choice sets.

Case 2: α has a subinterval of ordertype $k \cdot \eta + m \cdot \eta$, where $k \neq m$. By Example 6.14 and Lemma 2.2, α does not admit computable choice sets.

Case 3: α has a subinterval of ordertype $1 + \delta + k \cdot \eta + 1$ or $(1 + \delta + k \cdot \eta + 1)^*$, where δ is not uniformly dense. By Lemmas 2.1, 2.2 and2.4, it suffices to consider the case in which α has ordertype $1 + \delta + k \cdot \eta + 1$. Given any finite separation of α, implement the small block strategy of Theorem 6.1 only on the interval I which has all of the following properties:

- I has a maximal block of cardinality k;
- I has a maximal block of cardinality $\neq k$.
- For every interval to the right of I and every maximal block B in that interval which does not contain a separation point, B has cardinality k.

I is clearly uniformly arithmetically definable; in fact, it is easy to obtain a Σ^0_3 definition of I. Hence by Theorem 6.1 relativized to an oracle of degree $\mathbf{0}'''$, there is a model of α of degree $\leq \mathbf{0}'''$ with no choice set of degree $\leq \mathbf{0}'''$.

Case 4: Otherwise. By Lemma 2.4, we may assume that $\alpha = m_0 + \delta_0$ for some integer m_0 and order-type δ_0 which does not have a smallest element. As Case 1 does not apply, we can decompose $\delta_0 = k_0 \cdot \eta + \xi_0$ for some integer k_0 where ξ_0 has no initial segment of order-type $k_0 \cdot \eta$. As Cases 2 and 3 do not apply, we can decompose $\xi_0 = m_1 + \delta_1$, for some integer m_1 and order-type δ_1 which does not have a smallest element. Continuing in this way inductively, we see that we can write $\alpha = (\sum_{i=0}^{n} m_i + k_i \cdot \eta) + \rho$, where for all i, it is not the case that $k_i = m_i = k_{i+1}$. As Case 1 does not apply, ρ has no least element, and as Case 2 does not apply, ρ^* is not uniformly dense. We wish to implement the small block strategy of Theorem 6.1 only on the interval I containing the cut separating $(\sum_{i=0}^{n} m_i + k_i \cdot \eta)$ from ρ. This interval is uniquely determined by the following properties:

- Every interval to the left of I has only finitely many separators;
- I has infinitely many separators.

I is clearly uniformly arithmetically definable; in fact, it is easy to obtain a Δ^0_7 definition of I. Hence by Theorem 6.1 relativized to an oracle of degree $\mathbf{0}^{(\mathbf{7})}$, there is a model of α of degree $\leq \mathbf{0}^{(\mathbf{7})}$ with no choice set of degree $\leq \mathbf{0}^{(\mathbf{7})}$. □

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