Harmonic Measure from Two Sides (and Tools from Geometric Measure Theory)

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Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain.

\[ \exists! \text{ family of probability measures } \{\omega^X\}_{X \in \Omega} \text{ on the boundary } \partial \Omega \]

\text{called harmonic measure of } \Omega \text{ with pole at } X \in \Omega \text{ such that}

\[ u(X) = \int_{\partial \Omega} f(Q) d\omega^X(Q) \]

is the solution of (D)
Dirichlet Problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain.

\[
\Delta u = 0 \text{ in } \Omega \\
u = f \text{ on } \partial \Omega
\]

$\Delta = \partial_{x_1}x_1 + \partial_{x_2}x_2 + \cdots + \partial_{x_n}x_n$

$\exists!$ family of probability measures $\{\omega^X\}_{X \in \Omega}$ on the boundary $\partial \Omega$ called harmonic measure of $\Omega$ with pole at $X \in \Omega$ such that

\[
u(X) = \int_{\partial \Omega} f(Q)d\omega^X(Q) \quad \text{is the solution of (D)}
Dirichlet Problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain.

Dirichlet Problem

\[
\begin{aligned}
\Delta u &= 0 \text{ in } \Omega \\
\Delta &= \partial_{x_1x_1} + \partial_{x_2x_2} + \cdots + \partial_{x_nx_n} \\
\end{aligned}
\]

\[
\exists! \text{ family of probability measures } \{\omega^X\}_{X \in \Omega} \text{ on the boundary } \partial \Omega \text{ called harmonic measure of } \Omega \text{ with pole at } X \in \Omega \text{ such that}
\]

\[
u(X) = \int_{\partial \Omega} f(Q) d\omega^X(Q) \quad \text{is the solution of (D)}
\]
Let $\Omega \subset \mathbb{R}^n$ be a domain of locally finite perimeter, with harmonic measure $\omega$ and surface measure $\sigma = \mathcal{H}^{n-1}|_{\partial \Omega}$.

If the Poisson kernel $\frac{d\omega}{d\sigma}$ is sufficiently regular, then how regular is the boundary $\partial \Omega$?
FBP 1 Results

- (Kinderlehrer and Nirenberg 1977) Let $\Omega \subset \mathbb{R}^n$ be of class $C^1$.

  1. $\log \frac{d\omega}{d\sigma} \in C^{1+m,\alpha}$ for $m \geq 0$, $\alpha \in (0, 1) \implies \partial \Omega$ is $C^{2+m,\alpha}$.
  2. $\log \frac{d\omega}{d\sigma} \in C^\infty \implies \partial \Omega$ is $C^\infty$
  3. $\log \frac{d\omega}{d\sigma}$ is real analytic $\implies \partial \Omega$ is real analytic.

- (Alt and Caffarelli 1981) Assume $\Omega \subset \mathbb{R}^n$ satisfies necessary “weak conditions” (that includes $C^1$ as a special case). Then: $\log \frac{d\omega}{d\sigma} \in C^{0,\alpha}$ for $\alpha > 0 \implies \partial \Omega$ is $C^{1,\beta}$, $\beta = \beta(\alpha) > 0$.

- (Jerison 1987) In Alt and Caffarelli’s Theorem, $\beta = \alpha$.

- (Jerison 1987) $\log \frac{d\omega}{d\sigma} \in C^0 \implies \partial \Omega$ is VMO$_1$.

- (Kenig and Toro 2003) Studied FBP 1 with $\log \frac{d\omega}{d\sigma} \in$ VMO.
Examples of NTA Domains

Smooth Domains

Lipschitz Domains

Quasispheres (e.g. snowflake)

Question: How should we measure regularity of harmonic measure on domains which do not have surface measure?
Free Boundary Problem 2

$\Omega \subset \mathbb{R}^n$ is a 2-sided domain if:

1. $\Omega^+ = \Omega$ is open and connected
2. $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ is open and connected
3. $\partial \Omega^+ = \partial \Omega^-$

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain, equipped with interior harmonic measure $\omega^+$ and exterior harmonic measure $\omega^-$.

If the two-sided kernel $\frac{d\omega^-}{d\omega^+}$ is sufficiently regular, then how regular is the boundary $\partial \Omega$?
An Unexpected Example

\[ \log \frac{d\omega^-}{d\omega^+} \text{ is smooth does not imply } \partial \Omega \text{ is smooth} \]

Figure: The zero set of the harmonic polynomial
\[ h(x, y, z) = x^2(y - z) + y^2(z - x) + z^2(x - y) - 10xyz \]

\[ \Omega^\pm = \{ h^\pm > 0 \} \text{ is a 2-sided domain, } \omega^+ = \omega^- \text{ (pole at infinity), } \log \frac{d\omega^-}{d\omega^+} \equiv 0 \text{ but } \partial \Omega^\pm = \{ h = 0 \} \text{ is not smooth at the origin.} \]
Structure Theorem for FBP 2

**Theorem (B)*** Assume $\Omega \subset \mathbb{R}^n$ is a 2-sided NTA domain, $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in VMO(d\omega^+)$ or $\log \frac{d\omega^-}{d\omega^+} \in C^0(\partial \Omega)$.

There exists $d \geq 1$ (depending on the NTA constants) such that $\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$.

1. Every **blow-up** of $\partial \Omega$ about a point $Q \in \Gamma_k$ is the **zero set** $h^{-1}(0)$ of a homogeneous harmonic polynomial $h$ of degree $k$ which separates $\mathbb{R}^n$ into two components.

2. The “flat points” $\Gamma_1$ is a dense open subset of $\partial \Omega$ with Hausdorff dimension $n - 1$.

3. The “singularities” $\Gamma_2 \cup \cdots \cup \Gamma_d$ have harmonic measure zero.
Ingredients in the Proof

1. FBP 2 was studied by Kenig and Toro (2006) who showed that blow-ups of $\partial \Omega$ are zero sets of harmonic polynomials.

   ▶ We show that only zero sets of \textit{homogeneous} harmonic polynomials appear as blow-ups.

   ▶ We show the degree of polynomials appearing in blow-ups is unique at every $Q \in \partial \Omega$. Hence $\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$.

   ▶ We study topology and size of the sets $\Gamma_k$.

2. To classify geometric blow-ups of the boundary, we study measure-theoretic blow-ups of $\omega^\pm$ (tangent measures).

3. To show $\Gamma_1$ is open, we study local flatness properties of the zero sets of harmonic polynomials.
Polynomial Harmonic Measures

$h : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, $\Delta h = 0$

$\Omega^+ = \{X : h(X) > 0\}, \Omega^- = \{X : h(X) < 0\}$

(i.e. $h^\pm$ is the Green function for $\Omega^\pm$)

The harmonic measure $\omega_h$ associated to $h$ is the harmonic measure of $\Omega^\pm$ with pole at infinity; i.e., for all $\varphi \in C^\infty_c(\mathbb{R}^n)$,

$$\int_{h^{-1}(0)} \varphi d\omega_h = - \int_{\partial \Omega^\pm} \varphi \frac{\partial h^\pm}{\partial \nu} d\sigma = \int_{\Omega^\pm} h^\pm \Delta \varphi$$

Two Collections of Measures Associated to Polynomials

$P_d = \{\omega_h : h$ harmonic polynomial of degree $\leq d\}$

$F_k = \{\omega_h : h$ homogenous harmonic polynomial of degree $= k\}$
Blow-ups of the Boundary $\leftrightarrow$ Tangent Measures of $\omega$

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain, let $Q \in \partial \Omega$ and let $r_i \downarrow 0$.

**Theorem:** (KT) There is subsequence of $r_i$ (which we relabel) and an unbounded 2-sided NTA domain $\Omega_\infty$ such that

- Blow-ups of Boundary at $Q$ Converge:
  \[
  \partial \Omega_i = \frac{\partial \Omega - Q}{r_i} \rightarrow \partial \Omega_\infty \quad \text{in Hausdorff metric}
  \]

- Blow-ups of Harmonic Measure at $Q$ Converge:
  \[
  \omega_{i}^\pm(E) = \frac{\omega^\pm(Q + r_i E)}{\omega^\pm(B(Q, r_i))}
  \]
  satisfy $\omega_i^\pm \rightharpoonup \omega^\pm_\infty$

where $\omega^\pm_\infty$ is the harmonic measure of $\Omega^\pm_\infty$ with pole at infinity.

Each blow-up $\omega^\pm_\infty$ is called a tangent measure of $\omega^\pm$ at $Q$. 
Tangent Measures of $\omega^\pm$ when $\omega^+ \ll \omega^- \ll \omega^+$

**Theorem** (Kenig and Toro)

If $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in VMO(d\omega^+)$, then $\text{Tan}(\omega^\pm, Q) \subset \mathcal{P}_d$.

**Goal:** Show $\text{Tan}(\omega^\pm, Q) \subset \mathcal{F}_k$

for some $1 \leq k = k(Q) \leq d$.

$$\mathcal{P}_d = \{ \omega_h : h \text{ harmonic polynomial of degree } \leq d \}$$

$$\mathcal{F}_k = \{ \omega_h : h \text{ homogenous harmonic of degree } = k \}$$
Tangent Measures of $\omega^\pm$ when $\omega^+ \ll \omega^- \ll \omega^+$

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**Goal:** Show $\text{Tan}(\omega^\pm, Q) \subset F_k$

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Cones of Measures

A collection $\mathcal{M}$ of non-zero Radon measures is a d-cone if it is preserved under scaling and dilation of $\mathbb{R}^n$:

1. If $\nu \in \mathcal{M}$ and $c > 0$, then $c\nu \in \mathcal{M}$.
2. If $\nu \in \mathcal{M}$ and $r > 0$, then $T_{0,r}\nu \in \mathcal{M}$. [$T_{0,r}(y) = y/r$]

Examples

- Tangent Measures: $\text{Tan}(\mu, x)$
- Polynomial Harmonic Measures: $\mathcal{P}_d$ and $\mathcal{F}_k$

Size of a Measure and Distance to a Cone

- Let $\psi$ be Radon measure on $\mathbb{R}^n$. The “size” of $\psi$ on $B(0, r)$ is $F_r(\psi) = \int_0^r \psi(B(0, s)) ds$.

- Let $\psi$ be a Radon measure on $\mathbb{R}^n$ and $\mathcal{M}$ a d-cone. There is a “distance” $d_r(\psi, \mathcal{M})$ from $\psi$ to $\mathcal{M}$ on $B(0, r)$ compatible with weak convergence of measures.
Connectedness of Tangent Measures

Let $\mathcal{F}$ and $\mathcal{M}$ be $d$-cones such that $\mathcal{F} \subset \mathcal{M}$. Assume that:

- $\mathcal{F}$ and $\mathcal{M}$ have compact bases ($\{\psi : F_1(\psi) = 1\}$),
- (Property P) There exists $\epsilon_0 > 0$ such that whenever $\mu \in \mathcal{M}$ and $d_r(\mu, \mathcal{F}) < \epsilon_0$ for all $r \geq r_0$ then $\mu \in \mathcal{F}$.

**Theorem**

If $\text{Tan}(\nu, x) \subset \mathcal{M}$ and $\text{Tan}(\nu, x) \cap \mathcal{F} \neq \emptyset$, then $\text{Tan}(\nu, x) \subset \mathcal{F}$.

**Key Point:** (Under technical hypotheses) If one tangent measure at a point belongs to $\mathcal{F}$ then all tangent measures belong to $\mathcal{F}$.

First proved in [P], the theorem was stated in this form in [KPT].

- Preiss used the theorem to show Radon measures in $\mathbb{R}^n$ with positive and finite $m$-density almost everywhere are $m$-rectifiable.
- Kenig, Preiss and Toro used the theorem to compute Hausdorff dimension of harmonic measure when $\omega^+ \ll \omega^- \ll \omega^+$. 
Checking the Hypotheses: Rate of Doubling

- If \( \omega \in \mathcal{F}_k \), then \( \omega(B(0, r)) = cr^{n+k-2} \) where \( c \) depends on \( n \), \( k \) and \( \|h\|_{L^1(S^{n-1})} \). Thus \( \mathcal{F}_k \) is uniformly doubling: if \( \omega \in \mathcal{F}_k \) then
  \[
  \frac{\omega(B(0, 2r))}{\omega(B(0, r))} = 2^{n+k-2} \quad \text{for all } r > 0
  \]
  independent of the associated polynomial \( h \).

**Lemma:** \( \mathcal{F}_k \) has compact basis for all \( k \geq 1 \).

- If \( \omega \in \mathcal{P}_d \) is associated to a polynomial of degree \( j \leq d \) (not necessarily homogeneous), then for all \( \tau > 1 \)
  \[
  \frac{\omega(B(0, \tau r))}{\omega(B(0, r))} \sim \tau^{n+j-2} \quad \text{as } r \to \infty.
  \]

**Theorem:** The comparison constant depends only on \( n \) and \( j! \)

**Corollary:** If \( d_r(\omega, \mathcal{F}_k) < \varepsilon_0(n, d) \ \forall \ r \geq r_0(\omega) \), then \( k = j \).
Checking the Hypotheses: Rate of Doubling

- If $\omega \in \mathcal{F}_k$, then $\omega(B(0, r)) = cr^{n+k-2}$ where $c$ depends on $n$, $k$ and $\|h\|_{L^1(S^{n-1})}$. Thus $\mathcal{F}_k$ is uniformly doubling: if $\omega \in \mathcal{F}_k$ then
  $$\frac{\omega(B(0, 2r))}{\omega(B(0, r))} = 2^{n+k-2} \quad \text{for all } r > 0$$
  independent of the associated polynomial $h$.

  **Lemma:** $\mathcal{F}_k$ has compact basis for all $k \geq 1$.

- If $\omega \in \mathcal{P}_d$ is associated to a polynomial of degree $j \leq d$ (not necessarily homogeneous), then for all $\tau > 1$
  $$\frac{\omega(B(0, \tau r))}{\omega(B(0, r))} \sim \tau^{n+j-2} \quad \text{as } r \to \infty.$$  

  **Theorem:** The comparison constant depends only on $n$ and $j$!

  **Corollary:** If $d_r(\omega, \mathcal{F}_k) < \varepsilon_0(n, d) \forall r \geq r_0(\omega)$, then $k = j$. 
Polynomial Blow-ups are Homogeneous

**Theorem (B)**

Let \( \Omega \) be a 2-sided NTA domain. If \( \text{Tan}(\omega^+, Q) \subset \mathcal{P}_d \), then \( \text{Tan}(\omega^\pm, Q) \subset \mathcal{F}_k \) for some \( 1 \leq k \leq d \).

**Steps in the Proof**

1. Since \( \text{Tan}(\omega^+, Q) \subset \mathcal{P}_d \), there is a smallest degree \( k \leq d \) such that \( \text{Tan}(\omega^+, Q) \cap \mathcal{P}_k \neq \emptyset \). Show that \( \text{Tan}(\omega^+, Q) \cap \mathcal{P}_k \subset \mathcal{F}_k \).

2. Let \( \mathcal{F} = \mathcal{F}_k \) and \( \mathcal{M} = \text{Tan}(\omega^+, Q) \cup \mathcal{F}_k \). By the previous slide the hypotheses of the connectedness theorem are satisfied. Therefore, \( \text{Tan}(\omega^+, Q) \subset \mathcal{F}_k \).
Open Questions

\[
\log \frac{d\omega^-}{d\omega^+} \in VMO(d\omega^+) \Rightarrow \partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d.
\]

1. Find an upper bound on dimension of the “singularities” \( \Gamma_2 \cup \cdots \cup \Gamma_d \). (Conjecture: \( \dim_H \leq n - 3 \))

2. (Higher Regularity) For example, if \( \log \frac{d\omega^-}{d\omega^+} \in C^{0,\alpha} \), then at \( Q \in \Gamma_k \) is \( \partial \Omega \) locally the \( C^{1,\alpha} \) image of the zero set of a harmonic polynomial of degree \( k \)?

3. (Rectifiability) Does \( \Gamma_1 = G \cup N \) where \( G \) is \( (n - 1) \)-rectifiable and \( \omega^\pm(N) = 0 \)?
   ▶ The answer is yes if one assumes that \( \partial \Omega \) has locally finite perimeter (Kenig-Preiss-Toro, B).

4. Find other applications of the connectedness of tangent measures.
REFERENCES


M. Badger, *Flat points in zero sets of harmonic polynomials and harmonic measure from two sides*, preprint. arXiv:1109.1427
APPENDIX
Distance from Measure to a Cone

Let \( L(r) = \{ f : \mathbb{R}^n \to \mathbb{R} \mid f \geq 0, \text{Lip} f \leq 1, \text{spt} f \subset B(0, r) \} \).

If \( \mu \) and \( \nu \) are two Radon measures in \( \mathbb{R}^n \) and \( r > 0 \), we set

\[
F_r(\mu, \nu) = \sup \left\{ \left| \int f d\mu - \int f d\nu \right| : f \in L(r) \right\}.
\]

When \( \nu = 0 \),

\[
F_r(\mu, 0) = \int_0^r \mu(B(0, s)) ds =: F_r(\mu).
\]

Note that \( \mu_i \rightharpoonup \mu \) if and only if \( \lim_{i \to \infty} F_r(\mu_i, \mu) = 0 \) for all \( r > 0 \).

If \( \psi \) is a Radon measure and \( \mathcal{M} \) is a d-cone, we define a scaled version of \( F_r \) as follows:

\[
d_r(\psi, \mathcal{M}) = \inf \left\{ F_r \left( \frac{\psi}{F_r(\psi)}, \mu \right) : \mu \in \mathcal{M} \text{ and } F_r(\mu) = 1 \right\},
\]

i.e., normalize \( \psi \) so \( F_r(\psi) = 1 \) \& then take distance to \( \mathcal{M} \) on \( B_r \).
Homogeneous Harmonic Polynomials – “Big Piece” Lemma

Let $h : \mathbb{R}^n \to \mathbb{R}$ be homogenous harmonic polynomial of degree $k$.

**Key Lemma (B):** There is a constant $\ell_{n,k} > 0$ with the following property. For all $t \in (0, 1)$,

$$
\mathcal{H}^{n-1}\{\theta \in S^{n-1} : |h(\theta)| \geq t\|h\|_{L^\infty(S^{n-1})}\} \geq \ell_{n,k}(1 - t)^{n-1}.
$$

**Interpretation:** If $h$ is homogeneous harmonic polynomial, then $h$ takes big values on a big piece of the unit sphere.
Bounds for $\omega(B(0, r))$ as $r \to \infty$

Given $h : \mathbb{R}^n \to \mathbb{R}$ harmonic polynomial of degree $d$, $h(0) = 0$,

$$h = h_d + h_{d-1} + \cdots + h_1$$

where $h_k$ is homogeneous harmonic polynomial of degree $k$. In polar coordinates,

$$h(r\theta) = r^d h_d(\theta) + r^{d-1} h_{d-1}(\theta) + \cdots + rh_1(\theta),$$

$$\frac{dh}{dr}(r\theta) = dr^{d-1} h_d(\theta) + (d-1)r^{d-2} h_{d-1}(\theta) + \cdots + h_1(\theta).$$

**Fact:** Recall $\Omega^+ = \{X : h(X) > 0\}$. For all $r > 0$,

$$\omega(B(0, r)) = \int_{\partial B(0, r) \cap \Omega^+} \frac{dh^+}{dr} d\sigma$$

The $r^{d-1} h_d(\theta)$ term dominates as $r \to \infty$. Upper bounds for $\omega(B_r)$ are easy. Use “Big Piece” Lemma for lower bounds.