Real Analysis Preliminary Exam, January 2017

Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on $\mathbb{R}^n$ is denoted by m or dx. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball centered at x with radius $r > 0$. We denote by $C_c(\mathbb{R}^n)$ the space of compactly supported continuous functions in $\mathbb{R}^n$.

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1. (15 points)
   (a) State and prove Hölder's inequality.
   (b) Let $f \in L^2(\mathbb{R}, m)$ and set $F(x) := \int_0^x f(t)\,dt$. Prove that there exists some constant $C \geq 0$ such that
   $$|F(x) - F(y)| \leq C\,|x - y|^{1/2}$$
   for all $x, y \in \mathbb{R}$.

2. (15 points) Prove or disprove three of the following statements.
   (a) If $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^n, m)$, then it converges a.e.
   (b) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions which converges in $L^\infty(\mathbb{R}^n, m)$, then it converges a.e.
   (c) If U is a subset of $\mathbb{R}^n$ whose boundary has outer Lebesgue measure 0, then U is Lebesgue measurable.
   (d) Let $(X, \mathcal{A}, \nu)$ be a measure space, and suppose that μ is a signed measure on $(X, \mathcal{A})$ satisfying $\mu \ll \nu$. If $\nu(A) = 0$ then $\mu^+(A) = \mu^-(A) = 0$ where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

3. (10 points) Let $g \in L^1(\mathbb{R}^n, m)$ such that
   $$\int g(x)\phi(x)\,dx = 0$$
   for all $\phi \in C_c(\mathbb{R}^n)$, then $g = 0$ a.e.

4. (10 points) Let f be a nonnegative measurable real function such that for all $n \geq 1$,
   $$\int \frac{n^2}{n^2 + x^2} f\left(x - \frac{1}{n}\right) dx \leq 1.$$
   Show that $f \in L^1(\mathbb{R}, m)$ and $\|f\|_1 \leq 1$.

5. (10 points) Let $f, g \in L^1(\mathbb{R}^n, m)$ be non-negative functions such that
   $$\liminf_{k\to\infty} \frac{\int_{B(x,1/k)} f(y)\,dy}{\int_{B(x,1/k)} g(y)\,dy} \leq 1$$
   for m-a.e. $x \in \mathbb{R}^n$. Show that $f \leq g$ a.e.

6. (10 points) Let $\{q_j : j = 1, \dots\}$ be an enumeration of the rational numbers. For $n \geq 1$, consider the functions
   $$f_n(x) = \sum_{j \leq n} \frac{2^{-j}}{\sqrt{|x - q_j|}} \mathbf{1}_{\mathbb{R}\setminus\{q_j\}}(x).$$
   (a) Prove that $f(x) := \lim_{n\to\infty} f_n(x)$ exists a.e. and belongs to $L^1(I, m)$ for any bounded interval I.
   (b) Show that for any constant M the set of points $\{x \in \mathbb{R} : f(x) \leq M\}$ does not contain any interval.