Measure and Integration Prelim, January 2011

1. Let $f : [0, 1] \to \mathbb{R}$ be bounded.

   (a) Show that the set where f is continuous is Lebesgue measurable (even if f is not Lebesgue measurable).

   (b) Show that if f is not continuous on a set of full Lebesgue measure, then f is not Riemann integrable.

   Hint: consider the standard partition of $[0, 1]$ into 2^n subintervals, and define $F_n(x)$ to be the sup of f over the interval containing x and define $f_n(x)$ to be the inf of f over this interval.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose that f is a measurable nonnegative function satisfying $\int f d\mu = 1$. Compute $\lim_{n\to\infty} \int n \log\left(1 + \left(\frac{f(x)}{n}\right)^{\alpha}\right) d\mu(x)$ in three different cases:

   (a) $0 < \alpha < 1$

   (b) $\alpha = 1$

   (c) $\alpha > 1$

   Justify your answer in each case.

   Hint: writing $n = n^{\alpha} n^{1-\alpha}$, and the inequalities $\log(1 + u) \leq u$ and $1 + u^{\alpha} \leq (1 + u)^{\alpha}$ for $u \geq 0, \alpha \geq 1$ may be useful.

3. (a) Suppose $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$, and $a, b \in (0, \infty)$. Prove that $ab \leq a^p/p + b^q/q$. Hint: it may help to write the inequality in terms of $s = p \log a$ and $t = q \log b$.

   (b) State and prove Hölder's inequality for $p, q \in (1, \infty)$. Hint: first show that it is sufficient to prove the case where $\|f\|_p = \|g\|_q = 1$, then use (a).

4. Let $f(x, y) \in L^1(Q)$ where $Q = [0, 1] \times [0, 1]$ is the unit square in $\mathbb{R}^2$. Suppose that for any continuous function $g(y)$ on $[0, 1]$ we know

$$\int f(x, y)g(y)\, dy = 0 \text{ for almost every } x \in [0, 1].$$

   Prove that $f = 0$ a.e. on Q.