1. (15 points) State and prove the Monotone Convergence Theorem.

2. (15 points)

   (a) Write down the definition of a Stieltjes measure on the real line $\mathbb{R}$.

   (b) Find all Stieltjes measures $\nu \neq 0$ on $\mathbb{R}$ with
   \[
   \int f \, d\nu = \left( \int f \, d\nu \right) \left( \int g \, d\nu \right)
   \]
   for all non-negative continuous functions $f$ and $g$.

3. (15 points) Prove or disprove three of the following statements.

   (a) If $(f_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions which converges in $L^1(\mathbb{R}, dx)$ then it converges in measure.

   (b) If $(f_n)_{n \in \mathbb{N}}$ is a sequence of integrable functions that converges almost everywhere on $[0, 1]$, then it converges in $L^1([0, 1], dx)$.

   (c) If $(f_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions that converges almost everywhere on $[0, 1]$, then it converges in $L^\infty([0, 1], dx)$.

   (d) If $(f_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions which converges in $L^1(\mathbb{R}, dx)$ then it converges almost everywhere.

4. (10 points) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be $\sigma$-finite measure spaces. Let $E$ be a measurable subset of $X \times Y$. Recall that the $x$-section of $E$ is the set
   \[\{y \in Y \mid (x, y) \in E\}\]
   and the $y$-section of $E$ is the set
   \[\{x \in X \mid (x, y) \in E\}\].

   Use Fubini’s theorem to prove that if the $x$-section of $E$ has $\nu$-measure 0 for $\mu$-almost every $x \in X$, then the $y$-section of $E$ has $\mu$-measure 0 for $\nu$-almost every $y \in Y$.

5. (10 points) Compute
   \[
   \lim_{n \to \infty} \int_{1/n}^{\infty} \frac{n^{3/2}y^{1/2} + y^{1/4}}{n^{2}y^{2} + n^{-1}} \, dy
   \]
   and justify all steps of your reasoning.

6. (10 points) Prove one of the following statements.

   (a) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are measurable then $fg$ is also measurable.

   (b) Let $\mu$ be a signed measure. A set $A$ is a null set with respect to $\mu$ if and only if $|\mu|(A) = 0$, where $|\mu| = \mu^+ + \mu^-$ is the total variation of $\mu$. 