MATH 313 August 2008 PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) **(a)** Consider the following classical interpolation problem.

- **Given** $n+1$ support points

$$(x_i, f_i) \qquad i = 0, \ldots, n; \qquad\qquad (x_i \neq x_j \quad for \quad i \neq j).$$

- **Find** a polynomial $P(x)$ whose degree does not exceed n such that

$$P(x_i) = f_i, \qquad i = 0, \ldots n.$$

Define the Vandermonde matrix, and then reformulate the above interpolation problem as a matrix problem of solving a linear system of equations with the Vandermonde coefficient matrix.

(b) Use the condition

$$x_i \neq x_j \quad for \quad i \neq j,$$

to prove that the Vandermonde matrix is nonsingular.

(c) Use the fact established in (b) to prove that the classical interpolation problem of (a) has a unique solution.

(d) Let $P_{i_0 i_1 \ldots i_k}(x)$ be the (unique) polynomial that interpolates at points

$$(x_{i_m}, f_{i_m}) \qquad m = 0, \ldots k.$$

Prove that there exists a unique coefficient $f_{i_0 \ldots i_k}$ such that

$$P_{i_0 \ldots i_k}(x) = P_{i_0 \ldots i_{k-1}} + f_{i_0 \ldots i_k}(x - x_{i_0})(x - x_{i_1}) \cdots (x - x_{i_{k-1}}).$$

(e) Prove the recursion:

$$f_{i_0 \ldots i_k} = \frac{f_{i_1 \ldots i_k} - f_{i_0 \ldots i_{k-1}}}{x_{i_k} - x_{i_0}}.$$

Q2) **(a)** Let $N = 2M + 1$ and consider

$$\Psi(x) = \frac{A_0}{2} + \sum_{h=1}^{M} (A_h \cos hx + B_h \sin hx) \tag{1}$$

and

$$p(x) = \beta_0 + \beta_1 e^{ix} + \beta_2 e^{2ix} + \ldots + \beta_{N-1} e^{(N-1)ix}$$

Assume that $\Psi(x)$ and $p(x)$ agree at the N points

$$x_k = 2\pi k/N, \qquad k = 0, 1, \ldots, N-1$$

i.e.,

$$\Psi(x_k) = p(x_k), \qquad k = 0, 1, \ldots, N-1.$$

Use the relation between e^{x_k} and $e^{x_{N-k}}$ to find the matrix S such that

$$\begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_M & B_M & \cdots & B_2 & B_1 \end{bmatrix} = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{N-1} \end{bmatrix} \cdot S \tag{2}$$

(b) Explain why the matrix S in (2) is invertible, and use the uniqueness of the interpolation polynomial to show that the trigonometric polynomial (1) satisfying

$$\Psi(x_k) = y_k, \quad y_k \in \mathbb{C}, \quad k = 0, ..., N-1. \tag{3}$$

is unique.

(c) Explain how to solve the trigonometric interpolation problem in (3) with the help of (2) via the inverse FFT (provide the definition for the DFT matrix).

Q3) The integration formulas of **Newton and Cotes** have the form

$$\int_a^b f(x)dx \approx h \sum_{i=0}^{n} f(a+ih) \cdot \alpha_i, \qquad h := \frac{b-a}{n},$$

and they are obtained (i.e., the specific values for $\{\alpha_i\}_{i=0}^n$ are obtained) as follows.

- The integrand $f(x)$ is replaced by a suitable interpolating polynomial $P(x)$.
- The value $\int_a^b P(x)dx$ is taken as an approximate value for $\int_a^b f(x)dx$.

(a) Formulate the **trapezoidal rule** and derive it as a special case of the Newton-Cotes formulas (Hint: consider the case when the interpolating polynomial has degree one).

(b) Formulate the **Simpson's rule** and derive it as a special case of the Newton-Cotes formulas (Hint: consider the case when the interpolating polynomial has degree two).

(c) Verify that the Simpson's rule is exact on $[-1, 1]$ for all polynomials whose degree does not exceed 3 (Hint: Prove it by verifying that it holds exactly for $f(x) = 1, \quad x, \quad x^2, \quad x^3$).

(d) Prove the following theorem. For $(n+1)$ pairwise distinct nodes $x_0, x_1, \ldots, x_n$ there exist a unique set of parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that the quadrature formula

$$\int_a^b f(x)dx \approx \frac{x_n - x_0}{n} \sum_{i=0}^{n} f(x_i) \cdot \alpha_i$$

is exact for all polynomials whose degree does not exceed n. (Hint: one way to prove it is based on the fact that Vandermonde matrix is nonsingular, see, e.g., item **(b)** of question Q1.)

Q4) **(a)** Use the fact that each norm $\|\cdot\|$ on $\mathbb{C}^n$ is uniformly continuous (no need to prove the latter fact, just formulate it as a specific inequality) to prove the following theorem.

All norms on $\mathbb{C}^n$ are equivalent in the following sense. For each pair of norms $p_1(x)$ and $p_2(x)$ there are positive constants m and M satisfying

$$mp_2(x) \leq p_1(x) \leq Mp_2(x)$$

for all x.

(b) Prove that if F is an $n \times n$ matrix with $\|F\| < 1$, then $(I+F)^{-1}$ exists and satisfies

$$\|(I+F)^{-1}\| \leq \frac{1}{1-\|F\|}.$$

(c) Let A be a nonsingular $n \times n$ matrix, $B = A(I+F)$, $\|F\| < 1$, and x and Δx be defined by

$$Ax = b, \qquad B(x + \Delta x) = b.$$

Use **(b)** to prove that

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\|F\|}{1-\|F\|}$$

as well as

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{cond(A)}{1 - cond(A)\frac{\|B-A\|}{\|A\|}} \cdot \frac{\|B-A\|}{\|A\|}$$

if

$$cond(A)\frac{\|B-A\|}{\|A\|} < 1.$$