5310 PRELIM

Introduction to Geometry and Topology

January 2013

You may use any result that has been proven in class, unless the question directly asks you to prove the result. Please, state the results you are using, and check that the assumptions are satisfied.

1. (a) A topological space X is *locally path connected* if for any point $x \in X$ and every neighborhood U of x there is a path connected neighborhood V of x so that $V \subset U$. Prove that a connected, locally path connected space is path connected.

   (b) Give an example of a path connected space that is not locally path connected. Justify your answer.

2. Let $\mathbb{RP}^n$ be the real projective space, that is, the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation: $(x_0, .., x_n) \sim (y_0, .., y_n)$ if there exists $\lambda \in \mathbb{R}$ such that

   $$(y_0, .., y_n) = \lambda (x_0, .., x_n).$$

   (a) Prove that the quotient map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is open.

   (b) Prove that $\mathbb{RP}^n$ is second countable, compact and Hausdorff.

3. Let $\mathbb{D}$ be the closed unit disk, and $\mathbb{S}^1$ the unit circle.

   (a) Prove that there is no retraction $\mathbb{D} \longrightarrow \mathbb{S}^1$.

   (b) Prove that every continuous map $f : \mathbb{D} \to \mathbb{D}$ has a fixed point.

4. (a) Find the fundamental group of the 2-dimensional sphere $\mathbb{S}^2$ with $n \geqslant 1$ points removed.

   (b) Find the fundamental group of the torus $\mathbb{T}^2$ with $n \geqslant 1$ points removed.

5. Let $p : (Y, y_0) \to (X, x_0)$ be a covering map, where Y is path connected. Let $H = p_* (\pi_1 (Y, y_0))$ be the image of the fundamental group of Y in $\pi_1 (X, x_0)$. Prove that there is a bijection between the set $\pi_1 (X, x_0) \diagup H$ of right cosets of H and the fiber $p^{-1} (x_0)$.