Topology Preliminary Exam Fall 2003

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1. (a) Define what it means for a topological space to be compact (in terms of coverings by open sets).

(b) Prove that X is compact if and only if every collection of closed sets in X with the finite intersection property has a nonvoid intersection.

2. Let X and Y be topological spaces and assume that $X \times Y$ has the product topology. Let $p : X \times Y \longrightarrow X$ be the projection. Prove or give a counter example for each statement:

(a) p is open.

(b) p is closed.

(c) If X and Y are both connected then $X \times Y$ is connected.

3. Let R be an equivalence relation on a topological space X and let $p : X \longrightarrow X/R$ denote the projection to the set of equivalence classes. There is the quotient topology $\mathcal{T}_Q$ on X/R defined by p. Let $\mathcal{T}$ be an arbitrary topology on X/R that satisfies the following property: Given any function $g : X/R \longrightarrow Y$, g is continuous (with respect to $\mathcal{T}$) if and only if the composition $g \circ p$ is continuous.

Must $\mathcal{T}$ be the quotient topology $\mathcal{T}_Q$? Prove or give a counter example.

4. Prove or give a counter example for each statement:

(a) A compact subspace A of a space X is closed in X.

(b) Let X be a compact space and Y be Hausdorff space. Every continuous map $g : X \longrightarrow Y$ is also a closed map.

5. Let $\{X_\alpha | \alpha \in J\}$ be an indexed family of topological spaces. Prove that $\mathrm{Cl}(\prod_{\alpha \in J} A_\alpha) = \prod_{\alpha \in J} \mathrm{Cl}(A_\alpha)$ in $\prod_{\alpha \in J} X_\alpha$.