Preliminary Exam in Complex Analysis
January 2013

Instructions
All assertions require written justification. In particular, state and verify the hypotheses of any theorems you use. In complex analysis the terms 'analytic' and 'holomorphic' are used interchangeably.

1. Suppose the function f is analytic in the unit disc Δ, $f(0) = 0$ and $|f(z)| < 1$ for all $z \in \Delta$. Show that the series $\sum_{n=1}^{\infty} f(z^n)$ converges to an analytic function g in Δ.

2. Let f_n be a sequence of entire functions converging uniformly on compact subsets of $\mathbb{C}$ to a function f. Furthermore, suppose that for each n the zeros of f_n lie on the real axis. Show that f is identically 0 or f has zeros only on the real axis.

3. Let g be a meromorphic function on the Riemann sphere $\hat{\mathbb{C}}$.

(a) Prove that g is a rational function.

(b) Suppose g has a removable singularity at ∞. Show that $\lim_{z \to \infty} g'(z) = 0$.

4. Use residues to evaluate the integral $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3 - 2\cos\theta}\, d\theta$.

5. Suppose f is an entire function and M is a positive real number. Prove that there is at most one connected component in the complement of the set $\{z \mid |f(z)| < M\}$.

6. Let P denote the set $\{z = re^{i\theta} \mid 0 < r < 1,\ -\pi/2 < \theta < \pi\}$. Explicitly describe a 1-1 conformal map of P onto the unit disc Δ.