Name: ______________________________

Math 5410 Prelim August , 2011

Choose 5 out of the 6 questions.

(1a) Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Establish the parallelogram law, i.e., for all $x, y \in H$, one has:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \ .$$

(1b) Let K be a non-empty closed convex set in H. Show that for any $x \in H$, there exists a unique point $y \in K$ such that

$$\|x - y\| = dist(x, K) \ .$$

(hint: use (a) to show a sequence $\{y_n\}$ that attains $dist(x, K)$ is a Cauchy sequence).

(1c) Let $x \in X$ and let y be the point of K closest to x as in (b). Prove that $\langle x - y, v - y \rangle \leq 0$ for all $v \in K$.

(2a) Find the Green's function $G(x, y)$ for the operator A where

$$Au = -u'' + u$$

with $u(0) = u'(1) = 0$.
(2b) Define $T : L^2(0,1) \to L^2(0,1)$ such that for any $f \in L^2(0,1)$,

$$(Tf)(x) = \int_0^1 G(x, y) f(y) \, dy \ .$$

Explain what spectral theorem is and why it is applicable.
(2c) Show that $\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of T}\}$.
(2d) Compute $\|T\|$. (hint: find eigenvalues of A).

(3) Let $k > 0$ and $u : \mathbf{R}^3 \setminus \{0\} \to \mathbf{R}$ defined by

$$u(x) \equiv -\frac{1}{4\pi|x|}e^{-k|x|} .$$

It is spherically symmetric, i.e. $u(x) = w(|x|)$ where $w : \mathbf{R} \to \mathbf{R}$ is given by $w(r) = -\frac{1}{4\pi r}e^{-kr}$.

(a) For a spherical symmetric function in $\mathbf{R}^3$, it is known that

$$\Delta u = [\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}]w .$$

Show that $(\Delta - k^2)u = 0$ for $x \neq 0$.

(b) Show that the distribution $\tilde{u}$ is a fundamental solution of the Helmholtz operator $\Delta - k^2$, i.e.

$$(\Delta - k^2)\tilde{u} = \delta .$$

(hint: the proof is similar to that for $k = 0$ when we deal with the Laplacian operator).

(4) Let H be a Hilbert space and $A : H \to H$ is compact.
(a) Give the definition that A is compact.

(b) Let $\{u_n\}$ be an orthonormal sequence in a Hilbert space and let $\{\lambda_n\}$ be a bounded sequence in $\mathbf{R}$. Prove that the operator $Ax = \sum \lambda_n \langle x, u_n\rangle u_n$ is compact if and only if $\lambda_n \to 0$ as $n \to \infty$.

(5a) Suppose f is an operator from a Banach space X to itself. Give the definition of f being Frechet differentiable at a point $x \in X$.

(5b) Show that the Frechet derivative, if exists, is unique.

(5c) Let $f : C[0,1] \to C[0,1]$ such that for any $g \in C[0,1]$, $t \in [0,1]$,

$$(f(g))(t) \equiv \left(\int_0^1 (g(\xi))^2 \, d\xi \right) \sin t \; .$$

Prove that f is differentiable at any g and find a formula for f'.

(5d) Given a bounded seqeunce $\{g_n\} \subset C[0,1]$, does $\{f(g_n)\}$ have a convergent subsequence? (i.e. is f a nonlinear compact operator?)

(6) Let

$$f(x) = \begin{cases} 1, & \text{if } |x| < \frac{1}{2}, \\ 0, & \text{otherwise} \end{cases}$$

Define $f_n(x) = nf(nx)$ for all $x \in \mathbf{R}$. Let $\tilde{f}_n$ be the distribution induced by f_n.

(a) Show that $\tilde{f}_n \to \delta$ in the distribution sense. Here δ is the delta function.

(b) Evaluate the distribution derivative $D\tilde{f}_n$ and find its limit.