Abstract Algebra Prelim Jan. 2010

1. Prove the rings $\mathbf{Z}/mn\mathbf{Z}$ and $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ are isomorphic when m and n are relatively prime (positive) integers. Discuss whether these rings are ever isomorphic when m and n are not relatively prime.

2. Let $S = \{(z, w) \in \mathbf{C} \times \mathbf{C} : |z|^2 + |w|^2 = 1\}$. For a positive integer m, let $\mathbf{Z}/m\mathbf{Z}$ act on the set S by
$$(a \bmod m) \cdot (z, w) = \left(e^{2\pi i a/m} z,\ e^{8\pi i a/m} w\right).$$
   (a) Show this is a group action of $\mathbf{Z}/m\mathbf{Z}$ on S.
   (b) If m is odd, show every orbit in this group action has m elements.
   (c) If m is even, show the orbit of some point in S has less than m elements.

3. Use Zorn's lemma to show every nontrivial finitely generated group contains a maximal subgroup. (A maximal subgroup is a proper subgroup contained in no other proper subgroup.) Do not assume the group is abelian.

4. (a) Let a be any complex number. Prove that the map $\phi\colon \mathbf{R}[x] \to \mathbf{C}$ defined by $\phi(f(x)) = f(a)$ is a homomorphism of rings.
   (b) Prove that $\mathbf{R}[x]/(x^2 + 1)$ is a field which is isomorphic to $\mathbf{C}$.

5. (a) Let R be a commutative ring with identity and I be an ideal in R. Show that R/I is a field if and only if I is a maximal ideal.
   (b) Let R be a PID and P be a nonzero prime ideal in R. Show that P is a maximal ideal.

6. Give examples as requested, with brief justification.
   (a) A nonabelian group which is not isomorphic to a semidirect product of nontrivial groups.
   (b) A 2-Sylow subgroup of S_4.
   (c) A PID other than $\mathbf{Z}$.
   (d) A unit other than ± 1 in $\mathbf{Z}[\sqrt{7}]$.