Abstract Algebra Prelim Jan. 2009

1. Let G be the subgroup of matrices in $\mathrm{GL}_2(\mathbf{R})$ of the form
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$
so $ad \neq 0$ and there are no constraints on b. Let G act on $\mathbf{R}^2$ in the usual way:
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ dy \end{pmatrix}.$$
   (a) Find the orbits of the points $\binom{0}{0}$, $\binom{1}{0}$, and $\binom{0}{1}$.
   (b) Compute the stabilizer subgroups in G of the points $\binom{0}{0}$, $\binom{1}{0}$, and $\binom{0}{1}$.
   (c) For g_1 and g_2 in $\mathrm{GL}_2(\mathbf{R})$, if $g_1 \cdot v = g_2 \cdot v$ for all $v \in \mathbf{R}^2$, does $g_1 = g_2$?

2. Let G be a group. Its commutator subgroup G' is the subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$, for all $x, y \in G$.
   (a) If H is a normal subgroup of G such that G/H is abelian, prove $G' \subset H$.
   (b) Show every subgroup H lying between G and G', i.e. $G' \subset H \subset G$, is a normal subgroup of G and G/H is abelian.

3. Let G and H be groups, $\varphi\colon H \to \mathrm{Aut}(G)$ a homomorphism.
   (a) Write down the group law in the semi-direct product $G \rtimes_\varphi H$ and determine the formula for the inverse of an element (g, h).
   (b) Show that the subset $\{(g, 1) : g \in G\}$ of $G \rtimes_\varphi H$ is a normal subgroup. What about the subset $\{(1, h) : h \in H\}$?
   (c) Explicitly define a homomorphism $\varphi\colon \mathbf{Z}/4\mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}/3\mathbf{Z})$ so that the semi-direct product $\mathbf{Z}/3\mathbf{Z} \rtimes_\varphi \mathbf{Z}/4\mathbf{Z}$ is nonabelian and give an example of two noncommuting elements in the group. (Of course for the additive groups $\mathbf{Z}/3\mathbf{Z}$ and $\mathbf{Z}/4\mathbf{Z}$, the identity is 0, not 1.)

4. (a) Find a generator for the ideal $(11 + i, 1 + 3i)$ in $\mathbf{Z}[i]$.
   (b) Find a generator for the ideal $(11 + i) \cap (1 + 3i)$ in $\mathbf{Z}[i]$. (Hint: how are generators of ideals (a, b) and $(a) \cap (b)$ in $\mathbf{Z}$ related?)

5. Let R be a commutative ring and S be a nonempty subset of R. The annihilator of S in R is the elements in R that multiply all of S to 0:
$$\mathrm{Ann}(S) = \{a \in R \mid ax = 0 \text{ for all } x \in S\}.$$
   (a) Show $\mathrm{Ann}(S)$ is an ideal in R.
   (b) Compute $\mathrm{Ann}(\{6, 9\})$ in $\mathbf{Z}/12\mathbf{Z}$.

6. Give examples as requested, with brief justification.
   (a) A group-theoretic property that distinguishes A_4 from D_6 (both have order 12).
   (b) A domain R and prime ideal $\mathfrak{p}$ such that R is *not* a PID but $R/\mathfrak{p}$ is a PID.
   (c) A ring R and an R-module that is not a free module.
   (d) A matrix $A \in \mathrm{M}_2(\mathbf{R})$ such that the only subspaces $V \subset \mathbf{R}^2$ for which $A(V) \subset V$ are $V = \{0\}$ and $V = \mathbf{R}^2$.