1. (10 pts)

(a) (5 pts) In a finite abelian group, prove the order of each element divides the maximal order of all elements. (You may use the classification of finite abelian groups.)

(b) (5 pts) In a field, use part a to prove every finite subgroup of $F^\times = F - \{0\}$ is cyclic.

2. (10 pts) Let $R$ be a commutative ring with identity and $R[x]$ be the polynomial ring over $R$.

(a) (3 pts) Prove the ideal $(x)$ in $R[x]$ is a prime ideal if and only if $R$ is an integral domain.

(b) (4 pts) Let $I$ be an ideal of $R$. Prove that the following set is an ideal in $R[x]$: 

$$I[x] := \{f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x] : a_0, a_1, \ldots, a_n \in I\}.$$ 

(c) (3 pts) Prove that an ideal $I$ of $R$ is a prime ideal if and only if the ideal $I[x]$ of $R[x]$ from part b is a prime ideal.

3. (10 pts) Let $G$ be a group and $H$ be a subgroup.

(a) (2 pts) Define the normalizer of $H$ in $G$.

(b) (4 pts) Prove conjugate subgroups have conjugate normalizers: if $N$ is the normalizer of $H$ in $G$, then for each $g \in G$, $gNg^{-1}$ is the normalizer of $gHg^{-1}$ in $G$.

(c) (4 pts) Let $G = \text{GL}_2(\mathbb{R})$ and $H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R}^\times \right\}$. Prove the normalizer of $H$ in $G$ is \[ \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R}^\times \right\}. \]

4. (10 pts) The Fibonacci numbers $\{f_n\}$ are determined recursively for $n \geq 0$ by $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 0$.

(a) (3 pts) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. For all integers $n \geq 1$, show $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$.

(b) (4 pts) Show the group $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$, for prime $p$, has order $(p^2 - 1)(p^2 - p)$.

(c) (3 pts) Use parts a and b to help you find, with proof, some integer $n \geq 1$ such that $f_n \equiv 0 \mod 10$ while $f_{n+1} \equiv 1 \mod 10$. (Hint: Use the prime factorization of 10.)

5. (10 pts) An abelian group $A$ is called divisible if, for each $a \in A$ and $n \in \mathbb{Z}^+$, there is a $b \in A$ (maybe not unique) such that $nb = a$. (Formally it says we can “divide” $a$ by $n$, but the choice may not be unique so do not write $b = \frac{1}{n}a$.) For example, $(\mathbb{R}, +)$ is divisible. Also $(\mathbb{C}^\times, \times)$ is divisible since, for all $n \in \mathbb{Z}^+$, a number in $\mathbb{C}^\times$ has an $n$th root in $\mathbb{C}^\times$ (not unique if $n > 1$).

Prove a nonzero divisible group can’t be finitely generated.

6. (10 pts) Give examples as requested, with justification.

(a) (2.5 pts) Two nonconjugate elements of $S_4$ that have the same order.

(b) (2.5 pts) Two commutative rings that are not isomorphic as rings but are isomorphic as additive groups.

(c) (2.5 pts) A formula for a ring isomorphism $\mathbb{R}[x]/(x^2 - 1) \to \mathbb{R} \times \mathbb{R}$.

(d) (2.5 pts) A cyclic $\mathbb{Z}[x]$-module (this means a $\mathbb{Z}[x]$-module having one generator) with a submodule that is not cyclic.