Abstract Algebra Prelim Aug. 2017

Justify all your steps. You may use any results that you know unless the question says otherwise, but don’t invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. Let $G$ be a nontrivial finite group. A subgroup $M$ is called a maximal subgroup of $G$ if $M$ is a proper subgroup, i.e., $M \neq G$, and the only subgroups of $G$ containing $M$ are $M$ and $G$.
   
   (a) Show each proper subgroup of a finite group $G$ is contained in a maximal subgroup of $G$.
   
   (b) Count the number of maximal subgroups of the dihedral group of order $2p$, where $p$ is an odd prime.
   
   (c) Show that if a nontrivial finite group $G$ has only one maximal subgroup then $G$ is cyclic of prime-power order. (Hint: first prove $G$ is cyclic.)

2. Let $G$ be a nonabelian group of order 75 and $H$ be a 5-Sylow subgroup of $G$.
   
   (a) Show $H$ is a normal subgroup of $G$ and is abelian.
   
   (b) Show $H$ is not cyclic, or equivalently $H \cong (\mathbb{Z}/5\mathbb{Z})^2$. (Hint: show the conjugation action of $G$ on $H$ is not trivial.)
   
   (c) Determine a $2 \times 2$ matrix $A$ with entries in $\mathbb{Z}/5\mathbb{Z}$ that has order 3. (Hint: you can find such a matrix with integer entries having complex eigenvalues equal to the primitive 3rd roots of unity $\zeta_3$ and $\zeta_3^2$, where $\zeta_3 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.)
   
   (d) Construct an example of a nonabelian group with order 75. (The matrix in part (c) will be useful.)

3. The group $S_9$ denotes the permutations of a set with 9 elements (symmetric group).
   
   (a) Prove that there is no element of order 18 in $S_9$.
   
   (b) Construct, with justification, an element of order 20 in $S_9$.

4. (a) Prove the only units in $\mathbb{Z}[\sqrt{-5}]$ are $\pm 1$.
   
   (b) Justify why the equation $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ shows $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain.
   
   (c) Justify why the equation $(2 + 3i)(2 - 3i) = (3 + 2i)(3 - 2i)$ does not show that $\mathbb{Z}[i]$ is not a unique factorization domain.

5. Let $R$ be a ring with identity, possibly noncommutative, and let $M$ be a left $R$-module. Denote by $\text{End}(M)$ the set of $R$-module endomorphisms of $M$, that is,
   
   $$ \text{End}(M) = \{ f : M \to M \mid f \text{ is an } R\text{-module homomorphism} \}.$$ 
   
   (a) Prove that $\text{End}(M)$ is a ring with identity under the operations
   
   $$(f + g)(m) = f(m) + g(m) \text{ and } (fg)(m) = (f \circ g)(m) \text{ for } f, g \in \text{End}(M), m \in M.$$ 
   
   (b) Suppose $R$ is a commutative ring with identity and $M = R$. Show the ring $\text{End}(R)$ is commutative.

6. Give examples as requested, with justification.
   
   (a) Two nonabelian groups of order 12 that are not isomorphic.
   
   (b) A group that acts transitively on the plane minus the origin, $\mathbb{R}^2 - \{ \mathbf{0} \}$.
   
   (c) A ring that is not a field and has infinitely many units.
   
   (d) A nonzero prime ideal in $\mathbb{R}[x, y]$ that is not a maximal ideal.