Algebra Preliminary Examination August 2001

1) Let $V$ be a finite dimensional vector space and $W_1, W_2$ subspaces of $V$. Show that

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

Here $W_1 + W_2 = \{w_1 + w_2 \mid w_i \in W_i\}$.

2) (i) Show that there does not exist a simple group of order 30.

(ii) Let $G$ be a finite group, $p$ a prime, and $P$ a normal $p$-Sylow subgroup of $G$. If $\phi : G \longrightarrow G$ is a homomorphism, show that $\phi(P) \subseteq P$.

3) Let $R$ be a commutative ring with identity. Call $R$ a $*$-ring if the intersection of all non-zero ideals of $R$ is non-zero. (The ring $R$ itself is an ideal.)

(i) Let $Z_n$ denote the ring of integers modulo $n$. Determine, with proof, those values of $n$ when $Z_n$ is a $*$-ring.

(ii) If $R$ is an integral domain $*$-ring, show that $R$ is a field.



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4) Let $A$ be a finitely generated infinite abelian group, and $n$ a positive integer. Show that there is a subgroup $B$ of $A$ with $|A/B| = n$.

5) Let $R$ and $S$ be integral domains with $R \subseteq S$. Suppose that $R$ is a PID. If $d$ is the greatest common divisor of $a$ and $b$ in $R$, show that $d$ is the greatest common divisor of $a$ and $b$ in $S$.

6) Let $R$ be a commutative ring with identity and consider the following commutative diagram of $R$-modules and $R$-module homomorphisms:

$$\begin{array}{ccccccccc} 0 & \rightarrow & A & \stackrel{\alpha}{\rightarrow} & B & \stackrel{\beta}{\rightarrow} & C & \rightarrow & 0 \\ & & \theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \\ 0 & \rightarrow & A' & \stackrel{\alpha'}{\rightarrow} & B' & \stackrel{\beta'}{\rightarrow} & C' & \rightarrow & 0 \end{array}$$

Here the rows are exact, meaning that $\alpha, \alpha'$ are monomorphisms, $\beta, \beta'$ are epimorphisms, $Im(\alpha) = Ker(\beta)$, and $Im(\alpha') = Ker(\beta')$.

Given that $\theta_1$ and $\theta_3$ are isomorphisms, show that $\theta_2$ is an isomorphism.