Name

Instructions. You have 2 hours for this exam. Write down your steps.

1. If \( \lim_{n \to \infty} a_n = a \), show that \( \lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a \).

2. Let \( f \) be a bounded function on \([a, b]\). Show that the functions defined by

\[
m(x) = \inf \{ f(\zeta) : \zeta \in [a, x] \}
\]

is continuous from the left on \((a, b)\).

3. Assume that \( f \in C([0, 2]) \) and \( f(0) = f(2) \). Prove that there exist \( x_1 \) and \( x_2 \) in \([0, 2]\) such that

\[
x_2 - x_1 = 1, \quad f(x_2) = f(x_1).
\]

4. Let \( f \) be continuous on \([0, 1]\) and differentiable on \((0, 1)\). Suppose \( f(0) = f(1) = 0 \) and there is \( x_0 \in (0, 1) \) such that \( f(x_0) = 1 \). Prove that \( |f'(c)| > 2 \) for some \( c \in (0, 1) \).