See-Saw Swap Solitaire and Other Games on Permutations

Tom ("sVen") Roby (UConn)

Joint research with Steve Linton, James Propp, & Julian West

Canada/USA Mathcamp
Lewis & Clark College
Portland, OR USA

29 July 2014

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html
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Outline

- The intermediary game
- Right superior game
- Typical results
- The general framework
- Table of results
- Open Problems
Who is sVen?

- Mathematician and educator at UConn (University of Connecticut),
- specializing in Combinatorics, Algebra, & Math Ed;
- Worked with programs for K-12 teachers, high-ability HS students, and ugrads who want help with math-intensive courses.
- Other interests include: folkdancing, Japanese culture & linguistics, Bulgarian singing, . . .
- More at http://www.math.uconn.edu/~troby
Shuffle the cards 1 through 6 and deal them out in a row.

3 1 4 6 5 2
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3 1 4 6 5 2

RULE: Interchange any two cards that have a card of intermediate rank lying (somewhere) between them.

GOAL: To put the cards in increasing order: 1, 2, 3, 4, 5, 6.
Shuffle the cards 1 through 6 and deal them out in a row.

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EG: May we interchange 4 and 2 above?
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EG: May we interchange 4 and 2 above? NO.

EG: May we interchange 3 and 6 above?
See-Saw Swqp Solitaire

Shuffle the cards 1 through 6 and deal them out in a row.

3 1 4 6 5 2

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GOAL: To put the cards in increasing order: 1, 2, 3, 4, 5, 6.

EG: May we interchange 4 and 2 above? NO.

EG: May we interchange 3 and 6 above? YES, to get

6 1 4 3 5 2
A Short Game

21\hat{3}456
A Short Game

21\hat{3}456
24\hat{3}1\hat{5}6
24\hat{3}651
A Short Game

21\hat{3}456
2431\hat{5}6
2\hat{4}3651
64\hat{3}251
A Short Game

\[ \underline{21} \hat{3} \underline{456} \]
\[ \underline{2431} \hat{5} \underline{6} \]
\[ \underline{243651} \]
\[ \underline{643251} \]
\[ \underline{143256} \]
A Short Game

21\hat{3}456

2431\hat{5}6

\hat{24}3651

64\hat{3}251

14\hat{3}256

123456
See-Saw Swap Solitaire Examples

213456  124356  314652

5 moves 436521 263451

136425 . . .

156423

153426

123456

11 moves
### See-Saw Swap Solitaire Examples

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>213456</td>
<td>124356</td>
<td>314652</td>
</tr>
<tr>
<td>243156</td>
<td>624351</td>
<td>514632</td>
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<tr>
<td>243651</td>
<td>654321</td>
<td>512634</td>
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<tr>
<td>643251</td>
<td>634521</td>
<td>542631</td>
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<tr>
<td>143256</td>
<td>631524</td>
<td>245631</td>
</tr>
<tr>
<td>123456</td>
<td>136524</td>
<td>265431</td>
</tr>
<tr>
<td>5 moves</td>
<td>436521</td>
<td>263451</td>
</tr>
<tr>
<td>436125</td>
<td>213456</td>
<td>...</td>
</tr>
<tr>
<td>136425</td>
<td>156423</td>
<td>...</td>
</tr>
<tr>
<td>153426</td>
<td>123456</td>
<td></td>
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<td>11 moves</td>
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See-Saw Swap Solitaire Examples

213456  124356  314652
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143256  631524  245631
123456  136524  265431

5 moves  436521  263451
436125  213456
136425  ... 
156423  12 moves
153426  
123456

11 moves
Lots of questions naturally arise:

1. Are the number of moves listed above shortest possible?

2. Can the player always win or is it possible to get stuck?

3. What’s the longest (# moves) an optimally played game can take?

4. What are good strategies or heuristics for winning?

5. Does the number of cards matter?

Who thinks the game gets easier with five cards? With seven?
One interesting case: Solve the Reverse Hand

7654321
How hard is reversing?

One interesting case: Solve the Reverse Hand

\[
\begin{align*}
\_765\hat{4}321 \\
165\hat{4}327
\end{align*}
\]
How hard is reversing?

One interesting case: Solve the Reverse Hand

765\hat{4}321
165\hat{4}327
125\hat{4}367
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\[
\begin{align*}
&765\hat{4}321 \\
&165\hat{4}327 \\
&125\hat{4}367 \\
&1234567 \\
\end{align*}
\]

3 moves
One interesting case: Solve the Reverse Hand

$\underline{7654321}$
$\underline{1654327}$
$\underline{1254367}$
$\underline{1234567}$

3 moves
How hard is reversing?

One interesting case: Solve the Reverse Hand

654321

7654321

1654327

1254367

1234567

3 moves
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One interesting case: Solve the Reverse Hand

\[
\begin{align*}
765\hat{4}321 & \quad 65\hat{4}321 \\
165\hat{4}327 \\
125\hat{4}367 \\
1234567 \\
3 \text{ moves}
\end{align*}
\]
How hard is reversing?

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3 moves
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One interesting case: Solve the Reverse Hand

7654321  654321
1654327  154326
1254367  124356
1234567

3 moves
How hard is reversing?

One interesting case: Solve the Reverse Hand

\[ 7654321 \quad 654321 \]
\[ 1654327 \quad 154326 \]
\[ 1254367 \quad 124356 \]
\[ 1234567 \quad \text{Now what?} \]

3 moves
How hard is reversing?

One interesting case: Solve the Reverse Hand

![Hand positions]

Now what?

3 moves

So for \( n \) odd, reversing is quick, taking \( \frac{n-1}{2} \) steps, but for \( n \) even it appears to be harder. Can one do better than \( 2 + 11 \) for \( n = 6 \)?
A permutation (on $n$) is an rearrangement of the numbers $1, 2, \ldots n$. We will think of permutations as “words” with no repeated letters made from this alphabet.
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For $n = 3$, there are six permutations: $123, 132, 213, 231, 312, 321$. 
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How many for $n = n$? $n!$

Let $S_n$ denote the set of all permutations on $n$. 
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How many for $n = n$? $n!$

Let $S_n$ denote the set of all permutations on $n$.

One can also think of permutations as being “bijective functions from \{1, 2, \ldots n\} to itself”, with a group structure given by composition of functions, but we don’t need that here.
So another way of expressing our game is as follows:

View permutations in $S_n$ as words: $a_1a_2\cdots a_n$, e.g., $314652 \in S_6$, and allow moves of the following type:

If $a_i < a_j < a_k$ or $a_i > a_j > a_k$ for some $i < j < k$, then we may interchange (swap) $a_i$ and $a_k$. 
See-Saw Swap Solitaire Examples

213456  124356  314652
243156  624351  514632
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436125  213456
136425  ... 
156423
153426
123456
We can think of this game as creating a graph, whose vertex set is $S_n$, and with edges between any two permutations connected by a legal move. We are interested in questions about the connected components of this graph.

From a slightly more high-falutin’ perspective, the transitive closure of the relations defined by these moves is an equivalence relation on the set of permutations. We are interested in the sizes and character of these equivalence classes.
What happens with our game in small cases?
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For $n = 3$ it’s clear that the only legal move is $123 \leftrightarrow 321$, so there are 5 distinct equivalence classes under this relation.
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For \( n = 4 \) we get ten equivalence classes including \{1234, 3214, 1432, 4321\}.
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For $n = 5$ we get 3 equivalence classes, and the class containing the identity contains 24 elements.

For $n \geq 6$ we get a single equivalence class (the “fully mixed” case), which of course contains the identity.
This leads to the following bar game:

- Demonstrate your skill at obtaining the identity from random permutations in $S_6$ using only the legal moves.
- Bet someone that they won’t be able to do it.
- When they get stuck (quite likely), take pity on them and give them a “easier” permutation in $S_5$.
- There’s an 80% chance that a randomly chosen $\sigma \in S_5$ is NOT legally obtainable by this set of moves.
This example illustrates the basic questions we will be interested in, not just for this game, but for ones with other sets of rules $P$:

A. Compute the number of equivalence classes $\#\text{Classes}^*(n, P)$ into which $S_n$ is partitioned.

B. Compute the size of $\#\text{Eq}^*(\iota_n, P)$ of the equivalence class containing the identity, $\iota_n$.

C. (More generally) characterise the set $\text{Eq}^*(\iota_n, P)$ of permutations equivalent to the identity.
This example illustrates the basic questions we will be interested in, not just for this game, but for ones with other sets of rules $P$:

A. Compute the number of equivalence classes $\#\text{Classes}^*(n, P)$ into which $S_n$ is partitioned.
   5,10,3,1,1,1,...

B. Compute the size of $\#\text{Eq}^*(\iota_n, P)$ of the equivalence class containing the identity, $\iota_n$.
   2,4,24,720,5040,40320,...

C. (More generally) characterise the set $\text{Eq}^*(\iota_n, P)$ of permutations equivalent to the identity.
   all permutations for $n \geq 6$...
Basic questions

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$$5, 10, 3, 1, 1, 1, \ldots$$

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$$2, 4, 24, 720, 5040, 40320, \ldots$$

C. (More generally) characterise the set $\text{Eq}^*(\iota_n, P)$ of permutations equivalent to the identity. all permutations for $n \geq 6$. . . .

So the relation given by $P = \{123 \leftrightarrow 321\}$ (with no adjacency constraints) is not a particularly interesting example from this standpoint.
The Right Superior Game

Say that two $n$-permutations are equivalent if they differ by an **adjacent** transposition $a_i a_{i+1} \leftrightarrow a_{i+1} a_i$, where both inequalities $a_i < a_{i+2}$ and $a_{i+1} < a_{i+2}$ hold.

$P_2^\downarrow = \{123 \leftrightarrow 213\}$. 
How many permutations are equivalent to the identity?

Try to figure this out for $n = 3, 4, 5!$
How many permutations are equivalent to the identity?

Try to figure this out for $n = 3, 4, 5$!

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Eq°($\iota_n$, $P$)</td>
<td>2</td>
<td>4</td>
<td>12</td>
<td>36</td>
<td>144</td>
<td>576</td>
<td>2880</td>
<td>14400</td>
</tr>
</tbody>
</table>

Do you see a pattern in these numbers?
How many permutations are equivalent to the identity?

Try to figure this out for $n = 3, 4, 5$!

<table>
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<th>4</th>
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Do you see a pattern in these numbers?

**Theorem 1.** For the Right Superior Game, the number of $n$-permutations in the equivalence class of the identity is

$$\lfloor n/2 \rfloor ! \lceil n/2 \rceil !$$
How many permutations are equivalent to the identity?

Try to figure this out for $n = 3, 4, 5$!

<table>
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<th>4</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>#(\text{Eq}^\circ(\iota_n, \pi))</td>
<td>2</td>
<td>4</td>
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Do you see a pattern in these numbers?

**Theorem 1.** For the Right Superior Game, the number of $n$-permutations in the equivalence class of the identity is

\[
\lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!
\]

i.e., for $n = 2r$, \(\#\{\pi : \pi \leftrightarrow 123\ldots n\} = r!r!\).

and for $n = 2r + 1$, \(\#\{\pi : \pi \leftrightarrow 123\ldots n\} = r!(r + 1)!\).
Proof that this is an upper bound:
The largest element must be in the rightmost position. (Why?)

This implies that the second-largest element must be in one of the three rightmost positions. (Why?)

This implies that the third-largest element . . .

Now, placing the elements from largest to smallest, we have the following number of choices for each placement:

\[ 1 \cdot 2 \cdot 3 \cdots \lfloor n/2 \rfloor \cdot \lfloor n/2 \rfloor \cdots 3 \cdot 2 \cdot 1 \]
Proof of equality.
It remains to show that all permutations meeting these constraints are in fact reachable.

Imagine a target permutation meeting the constraints. That is, the first element (even case) or first two elements (odd case) are less than $\lceil n/2 \rceil + 1$, the next two elements are less than $\lceil n/2 \rceil + 2$, etc.

Target: kgfOdiPahNQcbTRjUmSVWXelYZ

**Step one.** Advance all the “large” elements as far as they will go by rippling them forward:
Proof of Attainability

.............NOPQRSTUVWXYZ
.............N.OPQRSTUVWXYZ
.............NO.PQRSTUVWXYZ
.............NOP.QRSTUVWXYZ
.............NOPQ.RSTUVWXYZ
.............NOPQRSTUVWXYZ

Target: kgfOdiPahNQcbTRjUmSVWXe1YZ


Now observe that the “small” elements can be permuted freely while leaving the “large” elements in place.

\[ fRjs \mapsto jRfS. \]
Target: kgfOdiPhNQcbTRjUmSVWXelYZ


Now observe that the “small” elements can be permuted freely while leaving the “large” elements in place.

\[ fRjs \leftrightarrow fjRS \leftrightarrow jfRS \leftrightarrow jRfS. \]
Target: kgfOdiPahNQcbTRjUmSVWXe1YZ


Now observe that the “small” elements can be permuted freely while leaving the “large” elements in place.

\[ fRjs \leftrightarrow fjRS \leftrightarrow jfRS \leftrightarrow jRfS \].

**Step two.** Using this observation, move the correct element into the first position. (In the odd case, move the two correct elements into the first two positions.) Because the target permutation obeys the constraints, this element (or pair of elements) will be small compared with the fixed skeleton of large elements which is facilitating their movement.

kN.O.P.Q.R.S.T.U.V.W.X.Y.Z
Target: \textit{kgfOdiPahNQcbTRjUmSVWXelYZ}

Continue to place elements two at a time:

\begin{align*}
\text{kN.O.P.Q.R.S.T.U.V.W.X.Y.Z} \\
\text{kgfO.P.Q.R.S.T.U.V.W.X.Y.Z} \\
\text{kgfOdP.Q.R.S.T.U.V.W.X.Y.Z} \\
\text{kgfOdiPQ.R.S.T.U.V.W.X.Y.Z} \\
\text{kgfOdiPahR.S.T.U.V.W.X.Y.Z} \\
\text{kgfOdiPahNQcbTRjUmSVWXelYZ}
\end{align*}
As one might expect, this algorithm constructs some permutations efficiently, but not others. Construction of the “furthest” permutation 32145:

```
1 2 3 4 5
1 3 2 4 5
1 3 4 2 5
3 1 4 2 5  <-- build the skeleton
3 1 2 4 5  <-- 3 is already at front, so advance 2
3 2 1 4 5  <-- we’re there, stop!
```
An example where the algorithm is inefficient, 12435:

\[
\begin{align*}
1 & \quad 2 \quad 3 \quad 4 \quad 5 \\
1 & \quad 3 \quad 2 \quad 4 \quad 5 \\
1 & \quad 3 \quad 4 \quad 2 \quad 5 \\
3 & \quad 1 \quad 4 \quad 2 \quad 5 \quad \text{ <-- build the skeleton} \\
1 & \quad 3 \quad 4 \quad 2 \quad 5 \quad \text{ <-- advance 1 (it just came from there!)} \\
1 & \quad 3 \quad 2 \quad 4 \quad 5 \quad ) \\
1 & \quad 2 \quad 3 \quad 4 \quad 5 \quad ) \quad \text{ <-- 3-step procedure for advancing 2} \\
1 & \quad 2 \quad 4 \quad 3 \quad 5 \quad )
\end{align*}
\]
From: James Propp <jpropp@cs.uml.edu>
Date: Wed, 8 Jul 2009 17:07:07 -0400
Subject: two hundred and ten questions


I'd like to know the partition of n! determined by the transitive closure of each of the following seven relations on S_n:


The two most interesting numbers are probably the number of components and the size of the component containing the permutation 1,2,3,...,n.

I should say that I want this information for _three_ distinct interpretations of what "123 --> 213" means:
(a) In the narrowest sense, it could mean that if pi(i+1) = pi(i)+1 and pi(i+2) = pi(i)+2, then you can swap the values of pi(i) and pi(i+1).
(b) More broadly, it could mean that if pi(i) < pi(i+1) < pi(i+2), then you can swap the values of pi(i) and pi(i+1).
(c) More broadly still, it could mean that if pi(i) < pi(j) < pi(k) for i < j < k, then you can swap the values of pi(i) and pi(j).

Jim
Consider interchanges of subwords of “type” $\sigma_1 \leftrightarrow \sigma_2$, where $\sigma_i \in S_3$.

As Jim described, this can be taken in three sense: (a) both indices and values must be adjacent; (b) entries must be in adjacent positions; (c) unrestricted in value or position.

Restricting entries to be adjacent values (but not necessarily positions) is equivalent to (b) by the map that sends $\pi \rightarrow \pi^{-1}$.

In theory one could consider any of the $B(6) = 203$ partitions of $S_3$ as defining a relation (or three) of this type, although some of these will be trivially equivalent.

To keep the problem within bounds, we currently consider only sets of relations of the form $\iota_3 \leftrightarrow \sigma$, where $\sigma \in S_3$. Equivalently, these are partitions of $S_3$ with a single nontrivial block (containing $\iota_3$).
How many equivalences classes for each relation?

#Classes($n, P$)

<table>
<thead>
<tr>
<th>Transpositions</th>
<th>general</th>
<th>indices adjacent</th>
<th>indices &amp; values adjacent</th>
</tr>
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<tbody>
<tr>
<td>123 ↔ 132</td>
<td>[5, 14, 42, 132, 429] Catalan</td>
<td>[5, 16, 62, 284, 1507, 9104]</td>
<td>[5, 20, 102, 626, 4458, 36144]</td>
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<td>123 ↔ 213</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>123 ↔ 321</td>
<td>[5, 10, 3, 1, 1, 1] trivial</td>
<td>[5, 16, 60, 260, 1260, 67442]</td>
<td>[5, 20, 102, 626, 4458, 36144]</td>
</tr>
<tr>
<td>123 ↔ 321 ↔ 312</td>
<td>[4, 2, 1, 1, 1, 1] trivial</td>
<td>[4, 8, 14, 27, 68, 159, 496]</td>
<td>[4, 16, 84, 536, 3912, 32256]</td>
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<td>[3, 4, 5, 8, 11, 20, 29, 57]</td>
<td>[3, 13, 71, 470, 3497]</td>
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Size of class containing identity: $\#\text{Eq}^*(\iota, P)$

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<td>[2, 6, 24, 120, 720]</td>
<td>[2, 4, 12, 36, 144, 576, 2880]</td>
<td>[2, 3, 5, 8, 13, 21, 34, 55] Fibonacci numbers</td>
</tr>
<tr>
<td></td>
<td>(n-1)!</td>
<td>product of two factorials</td>
<td></td>
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<td>123 ↔ 213</td>
<td>[2, 4, 24, 720]</td>
<td>[2, 3, 6, 10, 20, 35, 70, 126]</td>
<td>[2, 3, 4, 6, 9, 13, 19, 28] A000930</td>
</tr>
<tr>
<td></td>
<td>trivial</td>
<td>central binomial coefficients</td>
<td></td>
</tr>
<tr>
<td>123 ↔ 321</td>
<td>[3, 13, 71, 461]</td>
<td>[3, 7, 35, 135, 945, 5193]</td>
<td>[3, 4, 8, 12, 21, 33, 55, 88] A052952</td>
</tr>
<tr>
<td></td>
<td>connected A003319</td>
<td>Chinese Monoid</td>
<td></td>
</tr>
<tr>
<td>123 ↔ 132 ↔ 213</td>
<td>[3, 23, 120, 720]</td>
<td>[3, 9, 54, 285, 2160, 15825]</td>
<td>[3, 5, 9, 17, 31, 57, 105, 193] tribonacci numbers A000213</td>
</tr>
<tr>
<td></td>
<td>trivial</td>
<td>proven for odd terms</td>
<td></td>
</tr>
<tr>
<td>123 ↔ 132 ↔ 321</td>
<td>[3, 23, 120, 720]</td>
<td>[4, 21, 116, 713, 5030]</td>
<td>[4, 6, 13, 23, 44, 80, 149, 273] tribonacci A000073 $-[n$ even$]$</td>
</tr>
<tr>
<td>123 ↔ 213 ↔ 321</td>
<td>trivial</td>
<td>$n!$—central Catalan</td>
<td></td>
</tr>
</tbody>
</table>
Further Work & Open Problems

We’ve just begun the more general study of these kinds of relations. Plenty of open problems remain, including:

1. Find formulae for the unknown data in the table.
2. Recall our initial “Intermediary in-between” rule, but in the adjacent context. We prove that

\[
\#\text{Eq}^{\downarrow}(e_n, \{123, 321\}) = \binom{n - 1}{\lceil(n - 1)/2\rceil}
\]

in a fairly indirect way. Is there a simple combinatorial proof?
3. Understand the structure of the graphs one gets on these relations. Are the (like Bruhat order in the unconstrained case) posets?
4. Is there a useful length (distance from the identity) function?
Answer more generally what the sizes of all the equivalence classes are, or whether there’s a simple way to characterize them (as insertion tableaux characterizes all permutations which are Knuth equivalent).

Consider more general relations, defined by partitioning $S_3$ in different ways (more general block structures or connecting non-transpositions. Or even using relations within $S_4$?

Pierrot, Rossin, & West (FPSAC 2011) handle the other case of including non-transpositions within a unique non-singleton block containing $\nu_3$ of a partition of $S_3$ (e.g., \{123, 231\}).
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Kuszmaul & Zhou consider the case of moves of adjacent elements generated by cyclic shifts, e.g., for $k = 5$ allowing replacements within

$$\{12345, 23451, 34512, 45123, 51234\},$$

and characterize the non-singleton classes induced in $S_n$.

In recent work Kuszmaul [3] finds answers for many of the “doubly-adjacent” cases for which we failed to find formulae. He also provides several interesting generalizations.

Consider more general relations, defined by partitioning $S_3$ in different ways (more general block structures or connecting non-transpositions. Or even using relations within $S_4$?
Kuszmaul started working in this area as a high-school student via the PRIMES program at MIT, working with Darij Grinberg. He’s gone on to do other prize-winner work, e.g., 3rd place at Intel 2014 (for a different project).
1. Can the player always win or is it possible to get stuck? Yes if you have at least six cards.

2. For an optimal player, what’s the maximum number of moves to win? Don’t know.

3. What are good strategies or heuristics for winning? Don’t know.

4. Does the number of cards matter? Yes! (More positions for seven, but for five...)

5. Are there interesting (more fun? better?) variations on this game?
References 1


Thanks for your attention!
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Any questions?