The Eleven Clocks Problem

Tom Roby (UConn)
Joint work with Jim Propp (UMass Lowell)

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Presented at G4G 11

Slides available at:
http://www.math.uconn.edu/~troby/research.html
[or google “Tom Roby” for links from my website]
Who is Tom Roby?

- Mathematician and educator, based at UConn
- Interests in Combinatorics, Algebra, & Math Ed;
- Worked intensively with programs for K–12 teachers, high-ability HS students, and ugrads.
- Other interests include: folkdancing, Japanese culture & linguistics, Bulgarian singing, ...
- More at http://www.math.uconn.edu/~troby
Imagine eleven clocks, each showing the time in a particular city.

On the hour, you compute the median (middle value of the list) of the times shown on all the clocks.

Repeat this eleven more times at one-hour intervals. What is the mean of the medians?
**An Example 1**

- On the hour, compute the **median** time shown on the clocks.
- Repeat this eleven more times at one-hour intervals. What is the mean of the **medians**?

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An Example 2

Reorganize the data so that the first hour is in increasing order.

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An Example 3

- Identify in red the median for each hour. Their mean is 6.5.

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Notes & Variants

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**Variation:** What if the some clocks are broken and run a bit faster or slower? Will that affect things? Will the mean of the medians still be predictable?
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This is still open.
Bulgarian Solitaire was a game popularized by Martin Gardner in a 1983 article in *Scientific American*. Here we play the game on a deck of any size.

**Bulgarian Solitaire**

Start with the cards in any collection of heaps. Remove one card from each heap and use them to make a new heap.

For convenience, we reorder the heaps to be decreasing by size. A heap disappears if it has only one card which gets removed.
Bulgarian solitaire on variable-sized decks

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E.g., for $n = 8$, two trajectories are

53 $\rightarrow$ 422 $\rightarrow$ 3311 $\rightarrow$ 322 $\rightarrow$ 422 $\rightarrow$ ...

and

62 $\rightarrow$ 521 $\rightarrow$ 431 $\rightarrow$ 332 $\rightarrow$ 3221 $\rightarrow$ 4211 $\rightarrow$ 431 $\rightarrow$ ...

(the new heaps are underlined).
Bulgarian solitaire: heap averages

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From each starting point, we eventually reach a cycle (since finitely many possibilities).

The mean number of heaps in each cycle will be the same from any starting point.
The more general phenomenon Jim Propp & I and several others are studying is this:

**Homomesy**

Find actions on sets of objects which preserve certain statistics on average, i.e., across any orbit of the action, the statistic should have the same value. We call this **homomesy**.

For example, in the problem above, the objects are “lists of times”, the action is “increment by one hour”, and the statistic is “median”.

But there are many other examples. Some of my favorites involve actions on certain subsets (called order ideals) of rectangular graphs (posets). These have been coded up by Mike La Croix into some wonderful applets.

\[
\begin{align*}
\text{Area} &= 0 \\
\text{Area} &= 3 \\
\text{Area} &= 6 \\
\text{Area} &= 4 \\
\text{Area} &= 2 \\
\frac{(0+3+6+4+2)}{5} &= 3
\end{align*}
\]

Promotion

\[
\frac{(1+4+2+5+3)}{5} = 3
\]
The community of people working on this kind of dynamical combinatorics has identified many examples of homomesy within combinatorics;


Better yet, google “Tom Roby” for links from my website (including this talk);

Please let us know if you encounter this phenomenon in your own work with puzzles, etc!

http://www.math.uconn.edu/~troby