Math 1131Q Section 10

Sections 4.4 & 4.5

Oct 28, 2010
**Recap**

**Concave Up:** We find the intervals on which \( f''(x) > 0 \)

**Concave Down:** We find the intervals on which \( f''(x) < 0 \)

**Inflection Point:** Points where concavity changes and \( f \) is continuous.

Inflection points only occur at points where \( f \) is continuous.
When graphing a function, we can divide the graph into regions of the following four types, draw them separately and then connect the pieces together.
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\[ f' < 0 \text{ and } f'' > 0 \quad \text{ and } \quad f' > 0 \text{ and } f'' > 0 \]

\[ f' < 0 \text{ and } f'' < 0 \quad \text{ and } \quad f' > 0 \text{ and } f'' < 0 \]
Sketch the graph. Identify where the function is increasing/decreasing, the intervals where it is concave up and down, and the local extreme points.
\[
h(x) = 3x^5 - 5x^3 + 3
\]

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**Graphing Checklist**

1. Compute \( h' \) & \( h'' \)
2. Find the critical points
3. Find intervals of increase, decrease, local extremes
$h(x) = 3x^5 - 5x^3 + 3$

Sketch the graph. Identify where the function is increasing/decreasing, the intervals where it is concave up and down, and the local extreme points.

**Graphing Checklist**

1. Compute $h'$ & $h''$
2. Find the critical points
3. Find intervals of increase, decrease, local extremes
4. Find concavity & inflection points
\[ h(x) = 3x^5 - 5x^3 + 3 \]

Sketch the graph. Identify where the function is increasing/decreasing, the intervals where it is concave up and down, and the local extreme points.

**Graphing Checklist**

1. Compute \( h' \) & \( h'' \)
2. Find the critical points
3. Find intervals of increase, decrease, local extremes
4. Find concavity & inflection points
5. Locate any vertical and horizontal asymptotes
6. When feasible, find \( x \)-intercepts and \( y \)-intercepts
\[ h(x) = 3x^5 - 5x^3 + 3 \]

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1. Compute \( h' \) & \( h'' \)
2. Find the critical points
3. Find intervals of increase, decrease, local extremes
4. Find concavity & inflection points
5. Locate any vertical and horizontal asymptotes
6. When feasible, find \( x \)-intercepts and \( y \)-intercepts
7. Sketch graph (draw the pieces on the intervals between the critical points and inflection points as suggested on the previous page).
\[ h(x) = 3x^5 - 5x^3 + 3 \]

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We begin by determining where the function \( h(x) = 3x^5 - 5x^3 + 3 \) is increasing. There are only three critical points: \( x = -1, x = 0, x = 1 \).

The only place \( h'(x) \) can change sign is at a critical point. We select test points to determine the signs in the intervals between critical points.
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The only place \( h'(x) \) can change sign is at a critical point. We select test points to determine the signs in the intervals between critical points.
Now we locate the intervals where the function is concave up and concave down.

$$h'(x) = 15x^4 - 15x^2$$

$$h''(x) = 60x^3 - 30x = 60x(x^2 - 1/2) = 60x(x - \sqrt{1/2})(x + \sqrt{1/2})$$

$$h''(x) = 0$$ when $$x = -\sqrt{1/2}, 0, \sqrt{1/2}$$. The sign of $$h''(x)$$ does not change on the intervals between these points:

$$(-\infty, -\sqrt{1/2}), \ (-\sqrt{1/2}, 0), \ (0, \sqrt{1/2}), \ (\sqrt{1/2}, \infty)$$

We choose test points in each interval:

$$x = -1, \ x = -1/2, \ x = 1/2, \ x = 1$$
Evaluating $h''$ at these test points we find

\[
\begin{align*}
    h''(-1) &= -30 & h''(-\frac{1}{2}) &= 7.5 & h''(\frac{1}{2}) &= -7.5 & h''(1) &= 30 \\
    h''(x) &< 0 & h''(x) &> 0 & h''(x) &< 0 & h''(x) &> 0
\end{align*}
\]

We need to put this together with our information about $h'$

\[
\begin{align*}
    h'(-2) &= 180 & h'(-\frac{1}{2}) &= -2.8 & h'(\frac{1}{2}) &= -2.8 & h'(2) &= 180 \\
    h'(x) &> 0 & h'(x) &< 0 & h'(x) &< 0 & h'(x) &> 0
\end{align*}
\]
When graphing a function, we can divide the graph into regions of the following four types, draw them separately and then connect the pieces together.

- $f' < 0$ and $f'' > 0$
- $f' > 0$ and $f'' > 0$
- $f' < 0$ and $f'' < 0$
- $f' > 0$ and $f'' < 0$
Shape of graph of $h(x)$
Putting the pieces together and plotting the critical points and inflection points, we get the graph of $h(x)$. 

\[ (-1, 5), (-0.707, 4.237), (0, 3), (0.707, 1.762), (1, 1) \]
Determine the point(s) on $y = x^2 + 1$ that are closest to $(0, 2)$.

**Optimization Problems**

**First steps**

1. Draw a picture and label the variables.
2. Express the quantity to be optimized in terms of the variables and any constraints on the variables.
3. Refine the expression for the quantity so that it is a function depending on only one input variable. Determine the domain.
4. Now use calculus to find the absolute maximum.
Determine the point(s) on \( y = x^2 + 1 \) that are closest to \((0, 2)\).

We need to minimize the distance between the point \((0, 2)\) and any point \((x, y)\) that is on the graph.
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The distance formula gives us the distance between the point $(0,2)$ and a point $(x,y)$ on the graph:

$$d = \sqrt{(x - 0)^2 + (y - 2)^2}$$
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$$d = \sqrt{(x - 0)^2 + (y - 2)^2}$$

The point that minimizes the distance will also minimize the square of the distance and it will be easier to use the square of the distance and minimize that.

$$D = d^2 = x^2 + (y - 2)^2$$
Determine the point(s) on $y = x^2 + 1$ that are closest to $(0, 2)$.

Minimize: $D = d^2 = x^2 + (y - 2)^2$

Constraints: $y = x^2 + 1$
Determine the point(s) on \( y = x^2 + 1 \) that are closest to \((0, 2)\).

Minimize: \( D = d^2 = x^2 + (y - 2)^2 \)

Constraints: \( y = x^2 + 1 \)

**Solution 1.** We use the constraint \( y = x^2 + 1 \) in the most obvious way. We already have the constraint solved for \( y \) so we plug \( x^2 + 1 \) in for \( y \) and get

Minimize: \( D = d^2 = x^2 + (x^2 - 1)^2 \)

We can now take the derivative of \( D \), find the critical points and then the absolute minimum value.
Minimize:

\[ D = d^2 = x^2 + (y - 2)^2 \]

**Solution 2.** Instead of using the constraint \( y = x^2 + 1 \) to replace \( y \) in the square of the distance we replace \( x \). The constraint \( x^2 = y - 1 \), gives us
Minimize:

\[ D = d^2 = x^2 + (y - 2)^2 \]

**Solution 2.** Instead of using the constraint \( y = x^2 + 1 \) to replace \( y \) in the square of the distance we replace \( x \). The constraint \( x^2 = y - 1 \), gives us

Minimize:  \[ D = y - 1 + (y - 2)^2 = y^2 - 3y + 3 \]

We compute the derivatives \( D' = 2y - 3 \) and \( D'' = 2 > 0 \)
Minimize:

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There is one critical point, \( y = 3/2 \), and since the second derivative is always positive we know that this point must give the absolute minimum.
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There is one critical point, \( y = 3/2 \), and since the second derivative is always positive we know that this point must give the absolute minimum.

\[ x^2 = y - 1 \implies x^2 = 1/2 \implies x = \pm \sqrt{1/2} \]

The points are \((-\sqrt{1/2}, 3/2) \) and \((\sqrt{1/2}, 3/2) \)
A printer needs to make a poster that will have a total area of 200 in$^2$ and will have 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom. What dimensions will give the largest printed area?

**Label:**  $h$ for height and $w$ for width

**Constraint:**  $200 = wh$

**Clicker Question**

Find a formula in terms of $w$ for the area function $A(w)$ that we want to maximize.

(a) $200 - \frac{400}{w} - 3.5w$

(b) $207 - \frac{400}{w} - 3.5w$

(c) $193 - \frac{400}{w} - 3.5w$

(d) $200 + \frac{400}{w} + 3.5w$
A printer needs to make a poster that will have a total area of 200 in² and will have 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom. What dimensions will give the largest printed area?

Printed Area: \( h - 3.5 \times w - 2 \)

Maximize: \( A = (w - 2)(h - 3.5) \)

Constraint: \( 200 = wh \)

\[ \Rightarrow h = \frac{200}{w} \]
A printer needs to make a poster that will have a total area of 200 in$^2$ and will have 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom. What dimensions will give the largest printed area?

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Maximize: \( A = (w - 2)(h - 3.5) \)

Constraint: \( 200 = wh \)

\[ \Rightarrow h = \frac{200}{w} \]

Maximize: \( A = (w - 2)\left(\frac{200}{w} - 3.5\right) = 200 - 3.5w - \frac{400}{w} + 7 \)

\[ A = 207 - 3.5w - \frac{400}{w} \]

Answer = b
\[ A = 207 - 3.5w - \frac{400}{w} \]

**Domain:** \( h \geq 3.5, \ w \geq 2 \)

\[ A' = -\frac{7}{2} + \frac{400}{w^2} = -\frac{7w^2}{2w^2} + \frac{800}{2w^2} \]
\[ A = 207 - 3.5w - \frac{400}{w} \]

**Domain:** \( h \geq 3.5, \ w \geq 2 \)

\[ A' = -\frac{7}{2} + \frac{400}{w^2} = -\frac{7w^2}{2w^2} + \frac{800}{2w^2} \]

**Critical points:** \( w = \pm \sqrt{\frac{800}{7}} \sim \pm 10.69 \)

**End points:** \( w = 2 \quad A(2) = 0 \)
\[ A = 207 - 3.5w - \frac{400}{w} \]

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**Critical points:** \( w = \pm \sqrt{\frac{800}{7}} \sim \pm 10.69 \)

**End points:** \( w = 2 \quad A(2) = 0 \)

\( A' > 0 \) on \( (2, \sqrt{\frac{800}{7}}) \) and \( A' < 0 \) on \( (\sqrt{\frac{800}{7}}, \infty) \)

Absolute maximum value when \( w = \sqrt{\frac{800}{7}} \)
A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

\[ t = \text{number of hours since 2:00 PM} \]

**Minimize:** \[ d = \sqrt{x^2 + y^2} \]

\[ D = d^2 = x^2 + y^2 \]

We need to express \( D \) in terms of \( t \).
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\[ D = d^2 = x^2 + y^2 \]

We need to express \( D \) in terms of \( t \).

The boat heading South has been traveling for \( t \) hours at a rate of 20 km per hour and so the distance \( x = 20t \).
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\[ D = d^2 = x^2 + y^2 \]

We need to express \( D \) in terms of \( t \).

The boat heading South has been traveling for \( t \) hours at a rate of 20 km per hour and so the distance \( x = 20t \).

The other boat will reach the dock after traveling another \( 1 - t \) hours at a rate of 15 km per hour and so the distance \( y = 15(1 - t) \).
\( t = \text{number of hours since 2:00 PM} \)

\( 0 \leq t \leq 1 \)

**Minimize:**

\[
D = d^2 = (15(1 - t))^2 + (20t)^2
\]

**Domain:** \( 0 \leq t \leq 1 \)
$t =$ number of hours since 2:00 PM

$0 \leq t \leq 1$

**Minimize:**

$D = d^2 = (15(1 - t))^2 + (20t)^2$

**Domain:** $0 \leq t \leq 1$

$D' = -2(15)^2(1 - t) + 2(20)^2t = 2((15)^2 + (20)^2)t - 2(15)^2 = 0$

$t = \frac{2(15)^2}{2((15)^2 + (20)^2)} = \frac{9}{25}$
$t = \text{number of hours since 2:00 PM}$

$0 \leq t \leq 1$

**Minimize:**

$D = d^2 = (15(1-t))^2 + (20t)^2$

**Domain:** $0 \leq t \leq 1$

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D' = -2(15)^2(1-t) + 2(20)^2t = 2((15)^2 + (20^2)t - 2(15)^2 = 0
\]

\[
t = \frac{2(15)^2}{2((15)^2 + (20^2)} = \frac{9}{25}
\]

$D' < 0$ on $(0, 9/25)$ and $D' > 0$ on $(9/25, 1)$. So we have an absolute minimum when $t = 9/25$. 

We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $10 per ft\(^2\) and the material used to build the sides cost $6/ft\(^2\). If the box must have a volume of 50 ft\(^3\) determine the dimensions that will minimize the cost to build the box.

Our goal is to minimize the cost of the materials subject to the constraint that the volume must be 50 ft\(^3\). The cost for each side is just the area of that side times the appropriate cost.
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Our goal is to minimize the cost of the materials subject to the constraint that the volume must be 50 ft\(^3\). The cost for each side is just the area of that side times the appropriate cost.

Minimize: \( C = 10(2lw) + 6(2wh + 2lh) = 60w^2 + 48wh \)

Constraint: \( 50 = lwh = 3w^2h \)
Minimize:

\[ C = 10(2lw) + 6(2wh + 2lh) = 60w^2 + 48wh \]

**Constraint:** \[ 50 = lwh = 3w^2h, \ w > 0 \]

From the constraint we have \( h = \frac{50}{3w^2} \) and putting this into the cost function \( C \) gives
Minimize:

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From the constraint we have \( h = \frac{50}{3w^2} \) and putting this into the cost function \( C \) gives

\[ C = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w} \]

\[ C' = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \]
Minimize:

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**Constraint:** \( 50 = lwh = 3w^2h, \ w > 0 \)

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\[ C = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w} \]

\[ C'' = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \]

\[ C''' = 120 + 1600w^{-3} \]

Critical points:

\[ \sqrt[3]{\frac{800}{120}} \approx 1.8821 \]

\( C'' > 0 \) so \( C \) is concave up and has an absolute minimum at its only critical point in \((0, \infty)\).
Minimize:

\[ C = 10(2lw) + 6(2wh + 2lh) = 60w^2 + 48wh \]

Constraint: \[ 50 = lwh = 3w^2h, \ w > 0 \]

From the constraint we have \( h = \frac{50}{3w^2} \) and putting this into the cost function \( C \) gives

\[ C = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w} \]

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\[ C'' = 120 + 1600w^{-3} \]

Critical points:

\[ \sqrt[3]{\frac{800}{120}} \approx 1.8821 \]

\( C'' > 0 \) so \( C \) is concave up and has an absolute minimum at its only critical point in \((0, \infty)\).
Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

Begin with a sketch.
Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

Begin with a sketch.

Maximize:

\[ A = (2x)(2y) = 4xy \]
Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

Begin with a sketch.

Maximize:
\[ A = (2x)(2y) = 4xy \]

Constraint: \[ x^2 + y^2 = 16 \]

\[ \Rightarrow y = \pm \sqrt{16 - x^2} \]

Since the point that we are looking at is in the first quadrant we know that \( y \) must be positive and so we can take the + part of this.
\[ A = 4x\sqrt{16 - x^2}, \quad 0 \leq x \leq 4 \]

\[ A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = \frac{64 - 8x^2}{\sqrt{16 - x^2}} \]
\[ A = 4x\sqrt{16 - x^2}, \quad 0 \leq x \leq 4 \]

\[ A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = \frac{64 - 8x^2}{\sqrt{16 - x^2}} \]

\[ A'(x) = \frac{8(8 - x^2)}{\sqrt{16 - x^2}} = \frac{8(\sqrt{8} - x)(\sqrt{8} + x)}{\sqrt{(4 - x)(4 + x)}} \]
\[ A = 4x\sqrt{16-x^2}, \quad 0 \leq x \leq 4 \]

\[ A'(x) = 4\sqrt{16-x^2} - \frac{4x^2}{\sqrt{16-x^2}} = \frac{64-8x^2}{\sqrt{16-x^2}} \]

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Critical points & end points: \( x = 0, \sqrt{8} = 2\sqrt{2}, 4 \)

\[ A(0) = 0, A(2\sqrt{2}) = 32, A(4) = 0 \]
\[ A = 4x\sqrt{16 - x^2}, \quad 0 \leq x \leq 4 \]

\[ A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = \frac{64 - 8x^2}{\sqrt{16 - x^2}} \]

\[ A'(x) = \frac{8(8 - x^2)}{\sqrt{16 - x^2}} = \frac{8(\sqrt{8} - x)(\sqrt{8} + x)}{\sqrt{(4 - x)(4 + x)}} \]

**Critical points & end points:** \( x = 0, \sqrt{8} = 2\sqrt{2}, 4 \)

\( A(0) = 0, A(2\sqrt{2}) = 32, A(4) = 0 \)

Absolute maximum: \( A(2\sqrt{2}) = 32 \)
Clicker Question

A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 meters of framing materials what must the dimensions of the window be to let in the most light?

Constraint: \(12 = 2h + 2r + \pi r\)

Write a formula for the area function to be maximized.

Maximize: \(A(r) = \)

(a) \(12r - \pi + (\pi/2 - 2)r^2\)   (b) \(12r + \pi - (\pi/2 - 2)r^2\)
(c) \(12r - (2 + \pi/2)r^2\)   (d) \(10r - (2 + \pi/2)r^2\)   (e) None of these
Maximize:

\[ A = 2hr + \frac{1}{2}\pi r^2 \]

**Constraint:** \[ 12 = 2h + 2r + \pi r \]

\[ h = 6 - r - \frac{1}{2}\pi r \]
Maximize:

\[ A = 2hr + \frac{1}{2}\pi r^2 \]

Constraint: \[ 12 = 2h + 2r + \pi r \]

\[ h = 6 - r - \frac{1}{2}\pi r \]

\[ A = 2r(6 - r - \frac{1}{2}\pi r) + \frac{1}{2}\pi r^2 \]

\[ A = 12r - 2r^2 - \pi + \frac{1}{2}\pi r^2 \]

\[ A = 12r - \pi + (\frac{\pi}{2} - 2)r^2 \]

Answer = a