(1) Consider the matrix \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 10 \\ 3 & 10 & 7 & 24 \\ 5 & 16 & 1 & 61 \end{bmatrix} \), the vector \( b = \begin{bmatrix} 5 \\ 3 \\ -2 \\ 4 \end{bmatrix} \) and the vector equation \( Ax = b \).

(a) Solve the equation by reducing the augmented matrix to echelon form.

(b) Solve the equation by reducing the augmented matrix to reduced echelon form.

(c) Solve the equation using the \( LU \) factorization for \( A \).

(d) Solve the equation by finding \( A^{-1} \) and letting \( x = A^{-1}b \).

(e) Find \( |A| \), the determinant of \( A \).

(f) Solve the equation using Cramer's Rule.

(g) Prove that the columns of \( A \) form a linearly independent set of vectors.

(h) Looking at the columns of \( A \) as a set of vectors, what is its span?

(2) Let \( u = <3, -1, 5> \), \( v = <1, -12, -5> \), \( w = <1, 2, 3> \).

(a) Determine whether the set \( \{u, v, w\} \) is linearly independent or linearly dependent.

(b) Describe \( \text{Span}\{u, v, w\} \) geometrically.

(3) Let \( u = <1, 1, 0> \), \( v = <1, 0, 1> \), \( w = <0, 1, 1> \).

(a) Determine whether the set \( \{u, v, w\} \) is linearly independent or linearly dependent.

(b) Describe \( \text{Span}\{u, v, w\} \) geometrically.

(4) Consider an arbitrary set \( \{u, v, w\} \) in \( \mathbb{R}^4 \). Prove that \( \text{Span}\{u, v, w\} \neq \mathbb{R}^4 \).

(5) Prove: If \( \alpha, \beta \) are both solutions of the matrix equation \( Ax = b \), then \( \alpha - \beta \) is a solution of the homogeneous vector equation \( Ax = 0 \).

(6) Let \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \). Derive the formula for \( |A| \) as we did in class, using the equivalence \( |A| = D(ae_1 + de_2 + ge_3, be_1 + ce_2 + he_3, ce_1 + fe_2 + ie_3) \) and the properties of \( D \).

(7) Prove: If \( A = LU \) is the \( LU \) factorization of a square matrix \( A \), then \( |A| = |U| \).
(8) Given a square \( nxn \) matrix \( A = (a_{i,j}) \), define \( N(A) = \max_j (\sum_{i=1}^n a_{i,j}) \). In other words, if one calculates the sums of the elements in each individual column of \( A \), \( N(A) \) is the largest of those sums. Prove that if all the entries of square matrices \( A \) and \( B \) are non-negative, the \( N(AB) \leq N(A)N(B) \).

(9) Solve the Leontief production equation for an economy with three sectors, given that
\[
C = \begin{bmatrix}
0.5 & 0.5 & 0.2 \\
0.4 & 0.2 & 0.1 \\
0.3 & 0.1 & 0.2
\end{bmatrix}
\]
is the consumption matrix and \( d = \begin{bmatrix} 50 \\ 30 \\ 60 \end{bmatrix} \) is the final demand.

You may use a symbolic manipulation package such as Maple or Mathematica to help answer this question.

(10) Let \( A \) be the matrix given by
\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 \\
1 & 3 & 9 & 27 \\
1 & 4 & 16 & 64
\end{bmatrix}
\]

(a) Calculate \( |A| \) by hand, using elementary row and column operations until you reach the point where \( |A| = k|B| \) for some constant \( k \) and a matrix \( B \) which is either upper or lower triangular so that its determinant can be calculated by simply multiplying together the entries of its diagonal.

(b) What can you say about the linear independence or linear dependence of the columns of \( A \)?

(c) What can you say about the invertibility of \( A \)?