Stability and posets

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$RT_2^2$ and CAC

- $K_\omega$ is the (countably) infinite graph in which every pair of nodes is connected.
- $\overline{K}_\omega$ is the infinite graph in which no pair of nodes is connected.

**Theorem (Graph Version of Ramsey’s Theorem for Pairs ($RT_2^2$))**

Every infinite graph contains a copy of $K_\omega$ or $\overline{K}_\omega$.

**Theorem (Chain–Antichain (CAC))**

Every infinite poset has either an infinite chain or an infinite antichain.

In this talk, all chains and antichains are infinite.
For a poset $P$, define its *comparability graph* $G_P$ by

- domain of $G_P =$ domain of $P$
- $a$ and $b$ are connected in $G_P$ iff $a$ and $b$ are comparable in $P$

Then,

- copies of $K_\omega$ in $G_P$ are chains in $P$ (and vice versa)
- copies of $\overline{K}_\omega$ in $G_P$ are antichains in $P$ (and vice versa)

So, a solution to $RT_2^2$ in $G_P$ is a solution to $CAC$ in $P$. 

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How hard is it to solve CAC for a computable poset?

By transferring his results on $RT_2^2$, Jockusch proved

- In the arithmetic hierarchy: Every computable poset has a $\Delta_2^0$ chain, or a $\Delta_2^0$ antichain, or both a $\Pi_2^0$ chain and a $\Pi_2^0$ antichain.
- In low hierarchy: Every computable poset has a low$_2$ chain or antichain.

Herrmann proved that you cannot improve these bounds.

- There is a computable poset with no $\Sigma_2^0$ chains or antichains.
- There is a computable poset with no low chains or antichains.
A clever idea of Cholak, Jockusch and Slaman

Split $RT^2_2$ into a stable version $SRT^2_2$ and a cohesive version $CRT^2_2$.

**Definition**

$G$ is **stable** if for every $x \in G$, either $x$ is connected to almost every other node or $x$ is not connected to almost every node.

- $SRT^2_2$: Every infinite stable graph contains a copy of $K_\omega$ or $\overline{K_\omega}$.
- $CRT^2_2$: Every infinite graph has an infinite stable subgraph.
- $RT^2_2 \iff SRT^2_2 + CRT^2_2$
- $CRT^2_2$ is strictly weaker than $RT^2_2$
- Open question: Is $SRT^2_2$ strictly weaker than $RT^2_2$?
A clever idea of Hirschfeldt and Shore

Why not do the same thing for CAC?

To do this, they defined a notion of a stable poset (given later).

- **SCAC**: Every infinite stable poset has a chain or antichain.
- **CCAC**: Every infinite poset contains an infinite stable poset.
- **CAC ⇔ SCAC + CCAC**.
- Both SCAC and CCAC are strictly weaker than CAC.
- Analyzing SCAC and CCAC, they proved that CAC is strictly weaker than $RT^2_2$. 
Stable posets

Definition

Fix an infinite poset $P$. An element $a \in P$ is

- *small* if $a <_P b$ for almost all $b \in P$
- *large* if $b <_P a$ for almost all $b \in P$
- *isolated* if $a$ is incomparable with almost all $b \in P$

$S_P = \text{the set of small elements in } P$
$L_P = \text{the set of large elements in } P$
$I_P = \text{the set of isolated elements in } P$

Definition (Hirschfeldt and Shore)

A poset $P$ is *stable* if either $P = S_P \cup I_P$ or $P = L_P \cup I_P$. 
Our work

Why restrict to $P = S_P \cup I_P$ or $P = L_P \cup I_P$ in definition of stability?

**Definition**

An infinite poset is *weakly stable* if $P = S_P \cup L_P \cup I_P$.

Note that

$$\text{stable} \Rightarrow \text{weakly stable}$$

but not conversely. For example, let $P$ be the linear order $\omega + \omega^*$ viewed as a poset.

- $S_P =$ the elements in the $\omega$ part.
- $L_P =$ the elements in the $\omega^*$ part.
- $I_P = \emptyset$.

Therefore, $P$ is weakly stable but not stable.
Definition (Comparability graph $G_P$ of poset $P$)

$G_P = P$ with an edge between $a$ and $b$ if $a$ and $b$ are comparable.

$P$ is a weakly stable poset $\Rightarrow G_P$ is a stable graph

$P$ is a weakly stable poset $\not\Leftarrow G_P$ is a stable graph

For the linear order $\mathbb{Z}$ (viewed as a partial order), we have

- $G_{\mathbb{Z}} = K_\omega$ (and hence is a stable graph), but
- $S_{\mathbb{Z}} = L_{\mathbb{Z}} = I_{\mathbb{Z}} = \emptyset$ (and hence $\mathbb{Z}$ is not a weakly stable poset).

Notice that every copy $\mathcal{L}$ of $\mathbb{Z}$ has an infinite chain which is $\Delta^0_1(\mathcal{L})$. 
Theorem (JKLLS)

If an infinite poset has a copy \( P \) such that no chain is \( \Delta^0_1(P) \), then

\[
P \text{ is weakly stable } \iff G_P \text{ is stable}
\]

Assume \( G_P \) is stable but \( P \) is not weakly stable. Fix \( a \notin S_P \cup L_P \cup I_P \).

- \( a \notin I_P \) implies \( a \) is comparable with infinitely many (hence almost all) \( p \in P \).
- \( a \notin S_P \cup L_P \) implies there are infinitely many \( p > a \) and infinitely many \( p < a \).
- If \( b \leq a \), then \( b < p \) for infinitely many \( p \) and hence \( b \) is comparable with almost all \( p \in P \). (Same for \( b \geq a \).)
- Let \( X \subseteq P \) consisting of elements comparable to \( a \). \( X \) is \( \Delta^0_1(P) \).
- Every element of \( X \) is comparable with almost every \( p \in P \).
- There is a chain \( C \in \Delta^0_1(X) \) and hence \( C \in \Delta^0_1(P) \).
Reverse mathematics

These two notions of stability give rise to two different stable versions of CAC.

- **SCAC**: Every infinite *stable* poset has a chain or antichain.
- **WSCAC**: Every infinite *weakly stable* poset has a chain or antichain.

**Theorem (JKLLS)**

*Over RCA₀, SCAC and WSCAC are equivalent.*
Arithmetic hierarchy results

For a computable (weakly) stable $P$,

- each of $S_P$, $L_P$ and $I_P$ are $\Delta^0_2$
- if $P$ has chains, then $P$ has $\Delta^0_2$ chains
- if $P$ has antichains, then $P$ has $\Delta^0_2$ antichains

For stable posets, we can do better than $\Delta^0_2$.

**Theorem (JKLLS)**

Every computable stable poset has a computable chain or a $\Pi^0_1$ antichain.

However, the dual of this theorem fails.

**Theorem (JKLLS)**

There is a computable stable poset which has no $\Pi^0_1$ chain or computable antichain.
In the case of weakly stable posets, one cannot improve on $\Delta_2^0$.

**Theorem (JKLLS)**

There is a computable weakly stable poset which has no $\Pi_1^0$ chains or $\Pi_1^0$ antichains.
Lowness hierarchy

Theorem (Hirschfeldt and Shore)

*Every computable stable poset has a low chain or a computable antichain.*

The dual of this theorem does hold

Theorem (JKLLS)

*Every computable stable poset has a computable chain or a low antichain.*

and it can be generalized to weakly stable posets.

Theorem (JKLLS)

*Every computable weakly stable poset has a low chain or a computable antichain.*

The dual of this theorem is open: Does a computable weakly stable poset have a computable chain or a low antichain?