Problem 1. Determine whether the following sequences converge or diverge. If they converge, find their limit.

\[ a_n = \cos \frac{n\pi}{2} \]

The first sequence diverges because (starting with \( n = 0 \)) the values repeat in the pattern 1, 0, -1, 0.

\[ a_n = \frac{n^2 + 3n - 2}{5n^2} \]

The second sequence converges to 1/5. (To get this value, switch from \( n \) to \( x \) and use L'Hôpital’s Rule or the fact that it is a rational function in which the degrees of the numerator and the denominator are equal.)

\[ a_n = \frac{n^2}{n + 1} - \frac{n^2 + 1}{n} \]

To find the limit of the third sequence, rewrite it over a common denominator.

\[ a_n = \frac{n^2}{n + 1} - \frac{n^2 + 1}{n} = \frac{n^3 - (n^2 + 1)(n + 1)}{n(n + 1)} = \frac{n^2 + n + 1}{n^2 + n} \]

From here, you can see the limit is 1.

\[ a_n = 2^{1/n} \]

For the third sequence, as \( n \) approaches \( \infty \), the value of \( 1/n \) goes to 0. Therefore, the limit is \( 2^0 = 1 \).

\[ a_n = n^{(-1)^n} \]

The fourth sequence diverges because the absolute values of the terms go to infinity.

\[ a_n = \sqrt[n]{n} \]

For the last sequence, you can switch from \( n \) to \( x \) and use L'Hôpital’s Rule.

\[ \sqrt[n]{x} = x^{1/n} = e^\frac{\ln x}{1/x} = e^{\ln x/x} \]

By L'Hôpital’s Rule, \( \lim_{x \to \infty} \ln x/x = 0 \) and so the limit of the sequence is \( e^0 = 1 \).

Problem 2(a). Let \( \{a_n\} \) be a strictly decreasing sequence for which each term \( a_n > 0 \). Prove that \( \lim_{n \to \infty} a_n \geq 0 \).

Solution. There a couple of ways you might do this problem. They both start by noting that since \( \{a_n\} \) is a bounded monotonic sequence, then it has to converge. So, the sequence has a limit and the only question is to show that the limit is non-negative.

One method to show that \( \lim_{n \to \infty} a_n \geq 0 \) is to note that since \( \{a_n\} \) is a strictly decreasing sequence which is bounded below, we know it converges to \( \inf A \) where \( A \) is the set of numbers
in the sequence. Since 0 is a lower bound for $A$, we know that $\inf A \geq 0$ and therefore $\lim_{n \to \infty} a_n \geq 0$.

A second method to show $\lim_{n \to \infty} a_n \geq 0$ is by contradiction. Since we know that the sequence has a limit, we start by assuming that $\lim_{n \to \infty} a_n = c < 0$. We need to derive a contradiction. To get a contradiction, pick a value $\varepsilon$ such that $0 < \varepsilon < |c|$, and so $c + \varepsilon < 0$ (since $c$ is negative). Applying the definition of $\lim_{n \to \infty} = c$, there is an $N$ such that if $n \geq N$, then $|a_n - c| < \varepsilon$. Removing the absolute value signs gives

$$-\varepsilon < a_n - c < \varepsilon$$

and so

$$c - \varepsilon < a_n < c + \varepsilon$$

But, as we noted above, $c + \varepsilon < 0$ and so if $n \geq N$, then $a_n$ is negative! This contradicts the fact that each term in the sequence is positive.

2(b). Give a counterexample to show that you cannot in general conclude that $\lim_{n \to \infty} a_n > 0$.

**Solution.** Define a sequence by $a_n = 1/n$ for $n \geq 1$. The terms are positive and strictly decreasing, but $\lim_{n \to \infty} a_n = 0$.

**Problem 3.** Let $\sum_{k=0}^{\infty} a_k$ be a convergent series with positive terms. Prove that for every $\varepsilon > 0$, there is an $N$ such that if $n \geq N$, then $\sum_{k=n+1}^{\infty} a_k < \varepsilon$.

**Solution.** Since the given series converges, let $\sum_{k=0}^{\infty} a_k = S$. By definition, we know that the limit of the partial sums is equal to $S$. That is,

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k = S$$

Now, fix $\varepsilon > 0$ and we will try to find an appropriate $N$. Applying the definition of $\lim_{n \to \infty} \sum_{k=0}^{n} a_k = S$, there is an $N$ such that if $n \geq N$, then $|S - \sum_{k=0}^{n} a_k| < \varepsilon$. (Notice that I have written the difference inside the absolute values in a different order than usual. However, because of the absolute value signs, it doesn’t matter which order we subtract the terms in!) Removing the absolute value signs tells us that if $n \geq N$, then

$$-\varepsilon < S - \sum_{k=0}^{n} a_k < \varepsilon$$

But, $S = \sum_{k=0}^{\infty} a_k$ and so $S - \sum_{k=0}^{n} a_k = \sum_{n+1}^{\infty} a_k$. Therefore, we have that if $n \geq N$, then

$$-\varepsilon < \sum_{k=n+1}^{\infty} a_k < \varepsilon$$

and the right inequality is what we wanted to show.
Problem 4. Determine if the following telescoping series are convergent or divergent. If they converge, find the sum.

\[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \]

Solution. Use partial fractions to decompose the fraction.

\[ \frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} \]

which means

\[ 1 = A(2n+1) + B(2n-1) \]

Solving these gives \( A = 1/2 \) and \( B = -1/2 \). (One way to see this is to plug in \( n = 1/2 \) and \( n = -1/2 \).) Therefore, the \( n \)-th term looks like

\[ \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)} \]

The first few partial sums are

\[ s_1 = \frac{1}{2(1)} - \frac{1}{2(3)} \]
\[ s_2 = \left( \frac{1}{2(1)} - \frac{1}{2(3)} \right) + \left( \frac{1}{2(3)} - \frac{1}{2(5)} \right) = \frac{1}{2(1)} - \frac{1}{2(5)} \]
\[ s_3 = \left( \frac{1}{2(1)} - \frac{1}{2(5)} \right) + \left( \frac{1}{2(5)} - \frac{1}{2(7)} \right) = \frac{1}{2(1)} - \frac{1}{2(7)} \]

From here, the pattern emerges: \( s_n = 1/2 - 1/(2n+1) \). To find the value of the original series,

\[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim_{n \to \infty} s_n = \frac{1}{2} \]

Before going on to the second telescoping series, you might try to prove that \( s_n = 1/2 - 1/(2n+1) \) by induction on \( n \). I just stated it above because the pattern is fairly clear from the first few examples, but it a good exercise to prove it by induction. Use the fact that \( s_{n+1} = s_n + a_{n+1} \) and it should fall out relatively easily.

The second sum to consider is

\[ \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} \]

For this sum, the partial fraction decomposition is a longer. Since the denominator is a product of degree one factors, we write

\[ \frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2} \]
and therefore,
\[ 2n + 1 = An(n + 1)^2 + B(n + 1)^2 + Cn^2(n + 1) + Dn^2 \]

Plugging in \( n = 0 \) gives us that \( B = 1 \) and plugging in \( n = -1 \) gives us that \( D = -1 \). Therefore, we have
\[ 2n + 1 = A(n^3 + 2n^2 + n) + (n^2 + 2n + 1) + C(n^3 + n^2) - n^2 \]

Collecting terms of the same degree gives us
\[ 2n + 1 = (A + C)n^3 + (2A + 2C)n^2 + (A + 2)n + 1 \]

Comparing the \( n \) terms, we see that \( 2 = A + 2 \) and hence \( A = 0 \). Comparing the \( n^3 \) terms, we see that \( 0 = A + C \) and hence \( C = 0 \). Therefore, our partial fraction decomposition is
\[ \frac{2n + 1}{n^2(n + 1)^2} = \frac{1}{n^2} - \frac{1}{(n + 1)^2} \]

The first few partial sums are
\[ s_1 = \frac{1}{1^2} - \frac{1}{2^2} \]
\[ s_2 = \left( \frac{1}{1^2} - \frac{1}{2^2} \right) + \left( \frac{1}{2^2} - \frac{1}{3^2} \right) = \frac{1}{1^2} - \frac{1}{3^2} \]
\[ s_3 = \left( \frac{1}{1^2} - \frac{1}{2^2} \right) + \left( \frac{1}{2^2} - \frac{1}{3^2} \right) = \frac{1}{1^2} - \frac{1}{5^2} \]

The general pattern emerges that \( s_n = 1 - 1/(n + 1)^2 \). (As above, it is a good exercise to prove this formula by induction on \( n \).) Therefore, our sum is
\[ \sum_{n=1}^{\infty} \frac{2n + 1}{n^2(n + 1)^2} = \lim_{n \to \infty} 1 - \frac{1}{(n + 1)^2} = 1 \]

**Problem 5.** Determine if the following geometric series converge or diverge. If they converge, find the sum.
\[ \sum_{n=1}^{\infty} \frac{1 + 2^n}{3^n} = \sum_{n=1}^{\infty} (1/3)^n + \sum_{n=1}^{\infty} (2/3)^n = \left( \frac{1}{1 - 1/3} - 1 \right) + \left( \frac{1}{1 - 2/3} - 1 \right) \]
\[ \sum_{n=0}^{\infty} \frac{1}{2^{n/2}} = \sum_{n=0}^{\infty} (1/\sqrt{2})^n = \frac{1}{1 - 1/\sqrt{2}} \]
\[ \sum_{n=3}^{\infty} \frac{3^{n-1}}{e^n} = \frac{1}{3} \sum_{n=3}^{\infty} (3/e)^n \text{ which diverges since } 3/e > 1 \]
**Problem 6.** Let \( f(x) \) be a function which is continuous, strictly positive and strictly decreasing on the interval \([1, \infty)\) such that \( \int_1^\infty f(x) \, dx \) converges. By the Integral Test, the series \( \sum_{k=1}^\infty a_k \) with \( a_k = f(k) \) converges, so we have

\[
\sum_{k=1}^\infty a_k = S
\]

For this problem, you will give an error estimate using the Integral Test for this series. Let \( s_n = \sum_{k=1}^n a_k \) be the \( n \)-th partial sum for this series. The error in using \( s_n \) as an approximation to \( S \) is given by

\[
R_n = S - s_n = \sum_{n+1}^\infty a_k
\]

Draw a picture similar to the ones we used in the proof of the Integral Test to prove that \( R_n \leq \int_n^\infty f(x) \, dx \).

*I don't know how to draw this picture electronically but look at Figure 10.4 on page 397 to help you.*

**Problem 7(a).** By the Integral Test, we know that \( \sum_{k=1}^\infty 1/k^3 \) converges. Use Problem 6 to give an upper bound for the error in using \( \sum_{k=1}^{10} 1/k^3 \) to approximate the value of this series.

**7(b).** What is the least value of \( n \) for which \( s_n = \sum_{k=1}^n 1/k^3 \) is accurate approximation to within \( 5 \times 10^{-4} \)?

**Solution.** To do this problem, notice that the function we are working with is \( f(x) = 1/x^3 \). By Problem 6, we have that the error in using \( s_{10} \) to approximate the given series is less than \( \int_{10}^\infty 1/x^3 \, dx \). Therefore, we calculate

\[
\int_{10}^\infty x^{-3} \, dx = \lim_{t \to \infty} \int_{10}^t x^{-3} \, dx = \lim_{t \to \infty} -1/(2x^2) \big|_{10}^t = \lim_{t \to \infty} -1/(2t^2) + 1/200 = 1/200
\]

To do 7(b), we need to find the least \( n \) such that \( \int_n^\infty x^{-3} \, dx < 5 \times 10^{-4} \).

\[
\int_n^\infty x^{-3} \, dx = \lim_{t \to \infty} \int_n^t x^{-3} \, dx = \lim_{t \to \infty} -1/(2x^2) \big|_n^t = \lim_{t \to \infty} -1/(2t^2) + 1/(2n^2) = 1/(2n^2)
\]

Therefore, we need to find the least \( n \) such that \( 1/(2n^2) < 5 \times 10^{-4} \), which is the same as finding the least \( n \) such that \( 1/n^2 < 10^{-3} \). I'll leave you to find the exact value of \( n \).

**Problem 8.** Let \( f_n(x) = (\sin nx)/n \) and let \( f(x) = \lim_{n \to \infty} f_n(x) \) be the pointwise limit of the sequence of functions \( \{f_n(x)\} \). Show that \( f(x) \) is defined for all \( x \) and that \( f(x) = 0 \). Then show that

\[
\lim_{n \to \infty} f'_n(0) \neq f'(0)
\]

This example shows that limits of sequences of functions cannot always be interchanged with derivatives.
Solution. To show that \( f(x) = 0 \), we use the Squeeze Theorem. For any value of \( x \), we have
\[
\frac{-1}{n} \leq \frac{\sin nx}{n} \leq \frac{1}{n}
\]
Since \( \lim_{n \to \infty} 1/n = 0 \), the Squeeze Theorem tells us that \( \lim_{n \to \infty} (\sin nx)/n = 0 \).

To do the second part of the problem, we have \( f'_n(x) = \cos nx \) and therefore, \( f'_n(0) = 1 \) for each \( n \). This tells us that \( \lim_{n \to \infty} f'_n(0) = \lim_{n \to \infty} 1 = 1 \). On the other hand, \( f(x) = 0 \), so \( f'(x) = 0 \) and \( f'(0) = 0 \). Therefore, \( \lim_{n \to \infty} f'_n(0) = 1 \neq 0 = f'(0) \).

Problem 9(a). Prove the following integration formula for integers \( n \geq 1 \).
\[
\int_0^\pi \frac{\sin nx}{n^2} \, dx = \begin{cases} 
\frac{2}{n^3} & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even}
\end{cases}
\]

Solution. We calculate the integral by
\[
\int_0^\pi \frac{\sin nx}{n^2} \, dx = \left. -\frac{\cos nx}{n^3} \right|_0^\pi = -\frac{\cos n\pi}{n^3} - \frac{\cos 0}{n^3} = -\frac{\cos n\pi + 1}{n^3}
\]
Suppose \( n \) is odd. In this case, \( -\cos n\pi = -(-1) = 1 \) and the value of the integral is \( 2/n^3 \). On the other hand, if \( n \) is even, then \( -\cos n\pi = -1 \) and the value of the integral is \( 0 \).

9(b). Prove that the series \( \sum_{n=1}^\infty (\sin nx)/n^2 \) converges absolutely for all \( x \). Let \( f(x) \) denote the value of this sum.

Solution. To test for absolute convergence, we can take the absolute value and use the Comparison Test.
\[
0 \leq \left| \frac{\sin nx}{n^2} \right| \leq \frac{1}{n^2}
\]
Since the series \( \sum_{n=1}^\infty 1/n^2 \) converges, the Comparison Test tells us that \( \sum_{n=1}^\infty (|\sin nx|)/n^2 \) converges and hence the series \( \sum_{n=1}^\infty (\sin nx)/n^2 \) converges absolutely.

9(c). Use the Weierstrass M-Test to prove that the series of functions \( \sum_{n=1}^\infty (\sin nx)/n^2 \) converges uniformly to \( f(x) \).

Solution. To use the M-Test, we note that \( |\sin nx|/n^2 \leq 1/n^2 \) for all \( x \). Therefore, we can set \( M_n = 1/n^2 \). Since \( \sum_{n=1}^\infty M_n = \sum_{n=1}^\infty 1/n^2 \) converges, the M-Test tells us that \( \sum_{n=1}^\infty (\sin nx)/n^2 \) converges uniformly to its limit \( f(x) \).

9(d). Explain why \( f(x) \) is continuous (and hence integrable) on the interval \([0, \pi]\).

Solution. Since \( \sum_{n=1}^\infty (\sin nx)/n^2 \) converges uniformly to \( f(x) \) and since each function \( (\sin nx)/n \) is continuous, we know that the uniform limit function \( f(x) \) is also continuous.
9(e). Use the uniform convergence of $\sum_{n=1}^{\infty} (\sin nx)/n^2$ to $f(x)$ to integrate term-by-term and prove the formula

$$\int_0^{\pi} f(x) \, dx = \sum_{k=1}^{\infty} \frac{2}{(2k-1)^3}$$

**Solution.** To see the process of taking the term-by-term integration, it might be easier to write out our function.

$$f(x) = \frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \frac{\sin 5x}{5^2} + \cdots$$

Because of the uniform convergence, we have

$$\int_0^{\pi} f(x) \, dx = \int_0^{\pi} \frac{\sin x}{1^2} \, dx + \int_0^{\pi} \frac{\sin 2x}{2^2} \, dx + \int_0^{\pi} \frac{\sin 3x}{3^2} \, dx + \int_0^{\pi} \frac{\sin 4x}{4^2} \, dx + \int_0^{\pi} \frac{\sin 5x}{5^2} + \cdots$$

By this first part of this problem, the terms with even $n$ all have integrals equal to 0, so we have

$$\int_0^{\pi} f(x) \, dx = \int_0^{\pi} \frac{\sin x}{1^2} \, dx + \int_0^{\pi} \frac{\sin 3x}{3^2} \, dx + \int_0^{\pi} \frac{\sin 5x}{5^2} + \cdots$$

Again, using the first part, the integrals when $n$ is odd evaluate to $2/n^3$ so we have

$$\int_0^{\pi} f(x) \, dx = \frac{2}{1^3} + \frac{2}{3^3} + \frac{2}{5^3} + \cdots$$

We can write the sum on the right side as $\sum_{k=1}^{\infty} 2/(2k-1)^3$ using the fact that the terms $(2k-1)^3$ as $k$ ranges from 1 to $\infty$ give us exactly the terms $n^3$ for the odd numbers $n$ from 1 to $\infty$. 