Math 2142 Final Exam Review Problems

**Problem 1.** Prove the following properties of the cross product.

\[ A \times B = -(B \times A) \quad \text{and} \quad c(A \times B) = (cA) \times B \]

**Problem 2.** Let \( A \) and \( B \) be linearly independent vectors in \( \mathbb{R}^3 \). Prove that \( \{A, B, A \times B\} \) is a basis for \( \mathbb{R}^3 \).

*Hint.* Suppose \( aA + bB + c(A \times B) = \mathbf{0} \) for scalars \( a, b \) and \( c \). Take the dot product of each side with \( A \times B \). What does that tell you about \( c \)? How does that help you find \( a \) and \( b \)?

**Problem 3(a).** Let \( A = \langle a, b \rangle \) and \( B = \langle c, d \rangle \) be vectors in \( \mathbb{R}^2 \). Prove that \( A \) and \( B \) are linearly independent if and only if \( ad - bc \neq 0 \).

3(b). Let \( A \) and \( B \) are vectors in \( \mathbb{R}^2 \). Prove that \( \{A, B\} \) is a basis for \( \mathbb{R}^2 \) if and only if the matrix with rows \( A \) and \( B \) has a nonzero determinant.

**Problem 4(a).** Consider the plane through the point \((2, 0, 1)\) with direction vectors \( \langle 1, 1, 3 \rangle \) and \( \langle -4, 2, 1 \rangle \). Write the vector form of the equation for the plane.

4(b). Find the normal vector to the plane and give the normal form equation for the plane. Also, write the normal equation for the plane in the form \( ax + by + cz = d \). (The book calls this last equation the linear Cartesian equation for the plane.)

**Problem 5.** Determine whether the following sets of vectors form a basis for \( \mathbb{R}^2 \). If they do not form a basis, describe the span of the set of vectors.

\[
\begin{align*}
\{ \langle 1, 2 \rangle, \langle -2, -4 \rangle \} \\
\{ \langle 3, -2 \rangle, \langle 1, 4 \rangle \} \\
\{ \langle -1, 1 \rangle, \langle 0, 0 \rangle \}
\end{align*}
\]

**Problem 6.** Determine whether the following sets of vectors form a basis for \( \mathbb{R}^3 \). If they do not form a basis, find a nontrivial linear combination for \( \mathbf{0} \) and give a vector which is not in the span.

\[
\begin{align*}
\{ \langle 3, 0, 4 \rangle, \langle 2, 3, 2 \rangle, \langle 0, 5, -1 \rangle \} \\
\{ \langle 4, -2, -2 \rangle, \langle 3, -2, -3 \rangle, \langle -5, 4, 7 \rangle \} \\
\{ \langle 5, 0, 0 \rangle, \langle 7, 2, -6 \rangle, \langle 9, 4, -8 \rangle \}
\end{align*}
\]

**Problem 7.** Let \( F(t) \) and \( G(t) \) be differentiable vector valued functions. Prove that

\[
(F \cdot G)' = F' \cdot G + F \cdot G'
\]
Problem 8. Let $F(t)$ be a differentiable vector valued function and let $u(t)$ be a differentiable real valued (scalar) function. Let $G(t) = F(u(t))$ be the composition of these functions. Prove that

$$G'(t) = u'(t) F'(u(t))$$

Problem 9. Let $F(t)$ be a differentiable vector valued function such that $F(t)$ is never equal to $0$. Prove that

$$\|F(t)\|' = \frac{F(t) \cdot F'(t)}{\|F(t)\|}$$

*Hint.* You know $\|F(t)\| = \sqrt{F(t) \cdot F(t)}$.

Problem 10. Find the area of the triangle with vertices $P = (1, 2, 3)$, $Q = (2, 4, 5)$ and $R = (0, 3, 4)$.

*Hint.* Let $A$ be the vector from $P$ to $Q$ and let $B$ be the vector from $P$ to $R$. You know that $\|A \times B\|$ gives you the area of the parallelogram formed by these vectors. How is the area of the triangle related to the area of the parallelogram?

Problem 11(a). Let $A_1$ and $A_2$ be nonzero orthogonal vectors. Prove that $\{A_1, A_2\}$ is linearly independent.

*Hint.* Suppose $c_1 A_1 + c_2 A_2 = \overline{0}$. Take the dot product of both sides by $A_1$ and use the fact that $A_1$ and $A_2$ are orthogonal to help you show $c_1 = 0$.

Problem 12(a). Let $f(x)$ be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$ and let $C$ denote the curve which is the graph of $f$. Consider the vector valued function $F(t) = \langle t, f(t) \rangle$. Explain why $F(t)$ is a parameterization of $C$. Which direction along $C$ does a particle with position $F(t)$ move? Explain why $F(t)$ is a smooth parameterization.

12(b). Use $f(x)$ to give the usual Single Variable Calculus equation for the tangent line to $f(x)$ at $x = a$. Then use $F(t)$ to give the parametric form of the tangent line to $F(t)$ at $t = a$. Use some algebraic manipulations to show that these tangent lines are the same.
Exercises from textbook for practice.

- 12.4: 3, 8
- 12.8: 2, 4, 7, 8, 10
- 12.11: 2, 3, 11, 13
- 12.15: 1, 4, 13, 17
- 13.5: 5, 8
- 13.8: 3, 4, 5, 6
- 13.11: 1, 2, 3, 5
- 13.14: 1, 2, 3, 4
- 13.17: 1, 5(a)(b), 6
- 14.4: 1, 4, 7, 8, 10, 14, 15, 16, 18
- 14.7: 1-6, 8, 10
- 14.9: 1-6 (I didn’t check how bad the computations are), 7