Math 2141 Homework 3: Due Friday September 19

Problem 1. Prove that for all $n \in \mathbb{N}^+$, $n^3 + 5n + 6$ is divisible by 3.

Problem 2. Prove that for all $n \in \mathbb{N}^+$, $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$.

Problem 3. Prove by induction that for all $n \in \mathbb{N}$

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 4. Assume the following facts about the derivative:

- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$

Use induction to prove that for all $n \in \mathbb{N}^+$, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Problem 5. Let $p, n \in \mathbb{N}^+$. Prove that

$$n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p$$

Hint: You might be tempted to try proving this by induction, but there is a better way. First, notice that it is equivalent to prove

$$(p+1)n^p < (n+1)^{p+1} - n^{p+1} < (p+1)(n+1)^p$$

Try applying the Difference of Powers Formula from class. Write down this formula for the powers $a^{p+1} - b^{p+1}$. Then plug in $a = n+1$ and $b = n$. What is $a - b$? Count the number of terms remaining in the formula and think about the fact that $n^k < (n+1)^k$ for any power $k$.

Problem 6. Prove by induction on $n$ that

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p$$

Hint: Consider the two inequalities separately. That is, think of this as two separate problems and prove both inequalities by induction on $n$. Use Problem 5 to help you do the induction cases. Once you apply the inductive hypothesis, write down what you want to show and you will see why Problem 5 is helpful.

Practice Problems: If you want some extra practice problems you might try Section I.4.4 (page 35) Problems 1, 2, 3, 6 and Section I.4.7 (page 39) Problems 1, 2, 3, 4, 5, 6.