Problem 1. Prove that the following properties hold for all sets, $A, B,$ and $C$.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) $A - (B \cup C) = (A - B) \cap (A - C)$.

(c) If $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.

The test for set equality states that a set $X$ is equal to a set $Y$, or $X = Y$, if and only if $X \subseteq Y$ and $X \supseteq Y$. It is to be shown that this criteria is satisfied by the sets in (a) and (b).

(a) Let $X = A \cup (B \cap C)$ and $Y = (A \cup B) \cap (A \cup C)$.

"$\subseteq$" Suppose that $x \in X$. Then (i) $x \in A$ or (ii) $x \in B \cap C$.

(i) If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. Therefore, $x \in Y$.

(ii) If $x \in B \cap C$, then $x \in B$ and $x \in C$. Since $x \in B$, then $x \in A \cup B$. Since $x \in C$, then $x \in A \cup C$. Since $x \in A \cup B$ and $x \in A \cup C$, then $x \in Y$.

It has been shown by (i) and (ii) that every element in $X$ is also contained in $Y$. Thus, $X \subseteq Y$.

"$\supseteq$" Suppose that $x \in Y$. Then (i) $x \in A$ or (ii) $x \notin A$.

(i) $x \in A$. (This satisfies $x \in Y$ since $A \subseteq A \cup B$ and $A \subseteq A \cup C$ imply that $A \subseteq Y$.) This immediately implies that $x \in X$, since it is given that $A \subseteq X$.

(ii) $x \notin A$. Since $x \in Y$, then $x \in B \cap C$. Therefore, $x \in X$.

It has been shown by (i) and (ii) that every element in $Y$ is also contained in $X$. Thus, $X \supseteq Y$. Since $X \subseteq Y$ and $X \supseteq Y$, then $X = Y$, and thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.  

(b) Let $X = A - (B \cup C)$ and $Y = (A - B) \cap (A - C)$.

"$\subseteq$" Suppose that $x \in X$. Then $x \in A$ and $x \notin B \cup C$. This means that $x \notin B$ and $x \notin C$. By the definition of the difference of sets, $x \in A - B$ (since $x \in A$ and $x \notin B$) and $x \in A - C$ (since $x \in A$ and $x \notin C$). Therefore, $x \in Y$. Since all elements in $X$ are contained in $Y$, then $X \subseteq Y$.

"$\supseteq$" Suppose that $x \in Y$. Then $x \in A - B$ and $x \in A - C$. This means that $x \in A$ and $x \notin B$ and $x \notin C$. Since $x \notin B$ and $x \notin C$, then $x \notin B \cup C$. It has therefore been shown that $x \in A$ and $x \notin B \cup C$, and thus $x \in X$. Since all elements in $Y$ are also contained in $X$, then $X \supseteq Y$.

It has been shown that $X \subseteq Y$ and $X \supseteq Y$. This means that $X = Y$, and thus $A - (B \cup C) = (A - B) \cap (A - C)$.

(e) Suppose that $A \subseteq B \cup C$. This means that if $x \in A$, then $x \in B$ or $x \in C$. However, since $A \cap B = \emptyset$, then $x \notin B$. This only leaves $x \in C$. Since every member of $A$ is in $C$, then $A \subseteq C$.  

Problem 2. Let $A, B, C,$ and $D$ be sets. Prove that if $A \subseteq B$ and $C \subseteq D$, then $A - D \subseteq B - C$.

To show that $A - D \subseteq B - C$, we need to show that all elements of $A - D$ are also contained in $B - C$. More specifically, if $x \in A - D$, then $x \in B - C$.

Assume that $x \in A - D$. Then $x \in A$ and $x \notin D$. Since $x \in A$ and $A \subseteq B$ (given), then it follows that $x \in B$. Since $x \notin D$ and $C \subseteq D$ (also given), then it follows that $x \notin C$. Therefore, $x \in B$ and $x \notin C$, which means that $x \in B - C$.

It has been shown that all elements in $A - D$ are in $B - C$. Thus, $A - D \subseteq B - C$. □