1. (3 pts) Use the **ratio** test to determine whether or not the following series converges:

\[ \sum_{j=1}^{\infty} \frac{5^j}{j!} \]

**Solution:** We have that

\[ a_j = \frac{5^j}{j!} \text{ and } a_{j+1} = \frac{5^{j+1}}{(j+1)!}. \]

So we compute

\[
\lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j \to \infty} \frac{\frac{5^{j+1}}{(j+1)!}}{\frac{5^j}{j!}} \\
= \lim_{j \to \infty} \frac{5^{j+1} \cdot j!}{5^j \cdot (j+1)!} \\
= \lim_{j \to \infty} \frac{5 \cdot 5^j \cdot j!}{(j+1) \cdot j!} \\
= \lim_{j \to \infty} \frac{5}{j+1} \\
= 0 < 1.
\]

Since 0 is less than 1, then the series **converges**.

2. (3 pts) Use the **limit comparison** test to determine whether or not the following series converges:

\[ \sum_{k=1}^{\infty} \frac{k}{k^3 - 1} \]

**Solution:** To use the limit comparison we let \( a_k = \frac{k}{k^3 - 1} \) and \( b_k = \frac{1}{k^2} \). Noe we compute the following limit:

\[
\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{k}{k^3 - 1}}{\frac{1}{k^2}} \\
= \lim_{k \to \infty} \frac{k^3}{k^3 - 1} \\
= 1.
\]

Since 1 is a a positive number greater than zero, then either both \( a_k, b_k \) converge or diverge. But we now that \( \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2} \) is a \( p \)-series with \( p = 2 \). Since \( 2 > 1 \) then \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges. Hence

\[ \sum_{k=0}^{\infty} \frac{k}{k^3 - 1} \]

**converges.**
3. a. (3 pts) Use the alternating series test to determine whether or not the following series converges:

\[ \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \]

Solution:

We let \( b_m = \frac{1}{m} \). Clearly

\[ \lim_{m \to \infty} \frac{1}{m} = 0, \]

which is the first condition that we need to check. Next we have to check that \( b_m \) is decreasing. And it is since

\[ \frac{1}{m} > \frac{1}{m + 1} \]

for all \( m \). Another way to check decreasing is to compute the derivative, which in this case is \(-\frac{1}{m^2}\). But since this is always negative, then it means \( b_m = \frac{1}{m} \) have a negative slope, which means decreasing.

By the Alternating Series Test, then

\[ \sum_{m=1}^{\infty} \frac{(-1)^m}{m} = \sum_{m=1}^{\infty} (-1)^m b_m \]

is convergent.

b. (1 pt) Does \( \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \) converge absolutely? Justify your answer (briefly).

Solution:

The answer is NO. For if it did then the absolute value of the sequence would convergence as a series. But taking absolute values we get

\[ \sum_{m=1}^{\infty} \left| \frac{(-1)^m}{m} \right| = \sum_{m=1}^{\infty} \frac{1}{m}, \]

which is a Harmonic Series. Since Harmonic series diverges then the original series does NOT converge absolutely.