Solution to Homework 9

(3.3) Problem 4 We need to go back and solve (3.2) Problem 4
\[
\begin{align*}
\frac{dx}{dt} &= 5x + 4y \\
\frac{dy}{dt} &= 9x
\end{align*}
\]

\[
A = \begin{pmatrix} 5 & 4 \\ 9 & 0 \end{pmatrix}
\]

The characteristic polynomial of \(A\) is \(\lambda^2 - 5\lambda - 36\), Solving \(\lambda^2 - 5\lambda - 36 = 0\) gives \(\lambda_1 = 9\) and \(\lambda_2 = -4\). We can solve for the eigenvectors to be \(v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and \(v_2 = 4 - 9\).

The phase portrait will have two straight-lines: in the direction of \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\) (or \(\begin{pmatrix} -1 \\ -1 \end{pmatrix}\)), the arrows are pointing away from the origin; and in the direction of \(\begin{pmatrix} 4 \\ -9 \end{pmatrix}\) (or \(\begin{pmatrix} 4 \\ -9 \end{pmatrix}\)), the arrows are pointing towards the origin. Other curves start close to the direction of \(\begin{pmatrix} 4 \\ -9 \end{pmatrix}\) (or \(\begin{pmatrix} 4 \\ -9 \end{pmatrix}\)) and curve towards the direction of \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\) (or \(\begin{pmatrix} -1 \\ -1 \end{pmatrix}\)).

(3.4) Problem 22 This exercise is something that I hope to see once, but I didn’t expect you to discover it by yourself.

Consider the point \(\left(\frac{k_1}{\sqrt{k_1^2 + k_2^2}}, \frac{k_2}{\sqrt{k_1^2 + k_2^2}}\right)\), which lies on the unit circle. So there is an angle \(\phi\) such that
\[
\cos \phi = \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \quad \text{and} \quad \frac{k_2}{\sqrt{k_1^2 + k_2^2}}.
\]

Now we have
\[
x(t) = k_1 \cos \beta t + k_2 \sin \beta t
\]
\[
= \sqrt{k_1^2 + k_2^2} \cdot \left(\frac{k_1}{\sqrt{k_1^2 + k_2^2}} \cos \beta t + \frac{k_2}{\sqrt{k_1^2 + k_2^2}} \sin \beta t\right)
\]
\[
= \sqrt{k_1^2 + k_2^2} \cdot \left(\cos \phi \cos \beta t + \sin \phi \sin \beta t\right)
\]
\[
= \sqrt{k_1^2 + k_2^2} \cdot \cos(\beta t - \phi).
\]

(3.4) Problem 10

The characteristic polynomial is
\[
\lambda^2 - (2 + 6)\lambda + (2 \times 6 - 2 \times (-4)) = \lambda^2 - 8\lambda + 20 = 0
\]

We get \(\lambda = 4 \pm 2i\). For \(\lambda = 4 + 2i\), we solve
\[
\begin{pmatrix}
-2 - 2i & 2 \\
-4 & 2 - 2i
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

This has a solution \(\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+1 \\ i \end{pmatrix}\). So we have a complex solution
\[
Y(t) = e^{(4+2i)t} \begin{pmatrix} 1 \\ 1 + i \end{pmatrix} = \left( e^{4t} \cos 2t + ie^{4t} \sin 2t \right) \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}
\]
\[
= \left( e^{4t} \cos 2t - e^{4t} \sin 2t \right) + i \left( e^{4t} \cos 2t + e^{4t} \sin 2t \right).
\]

So the general solution is
\[
Y(t) = k_1 Y_{re}(t) + k_2 Y_{im}(t) = k_1 \left( e^{4t} \cos 2t - e^{4t} \sin 2t \right) + k_2 \left( e^{4t} \sin 2t \right)
\]

(3.5) Problem 20
Note that the eigenvalues $\lambda_1, \lambda_2$ are zeros of the characteristic polynomial of $A$, namely,

$$\lambda^2 - (a + d)\lambda + \det A = 0.$$ 

In particular, $\det A = \lambda_1 \lambda_2$. So if one or both $\lambda_i$ is zero, then $\det(A)$ being the product of $\lambda_1$ and $\lambda_2$ must be zero.

Conversely, if $\det(A) = 0$, then $\lambda_1 \lambda_2 = 0$; so at least one of $\lambda_i$ is zero.