Quadratic residues.
1. Use QR in $\mathbb{F}_2[T]$ to compute the symbols
   $$[T^3 + 1, T^4 + T^3 + 1], \quad [T^3 + 1, T^5 + T^3 + 1], \quad [T^5, T^7 + T^3 + T^2 + T + 1].$$
   (All moduli here are irreducible.)

2. Does the congruence $x^2 + (T^3 + T^3 + T)x \equiv T^8 + T + 1 \mod T^{11} + T^4 + T^2 + T + 1$ have a solution in $\mathbb{F}_2[T]$? (The modulus is irreducible.) Use QR in $\mathbb{F}_2[T]$ to answer the question.

3. (The result of this exercise is used in the proof of the quadratic reciprocity law in $\mathbb{F}_2[T]$.) Let $x_1, \ldots, x_d$ be variables and $p_m = x_1^n + \cdots + x_d^n$ be the $m$-th power sum, $m \geq 1$.
   Consider the identity
   $$(1 - x_1t)(1 - x_2t) \cdots (1 - x_dt) = 1 - s_1t + s_2t^2 - \cdots + (-1)^ds_dt^d,$$
   where $s_j$ is the $j$th elementary symmetric function of the $x$’s. Apply the operation $f(t) \mapsto f'(t)/f(t)$ to both sides, then expand them into formal power series ("generating functions") in $t$ to prove, for $1 \leq m \leq d$, that $p_m$ is an integral polynomial in $s_1, \ldots, s_m$. Make these formulas (also called the Newton identities) explicit for $m = 1, \ldots, 5$, over $\mathbb{Z}$ and then over $\mathbb{F}_2$.

   (The significance of the operation $f'/f$ is that it is log-like in $f$, converting products to sums, even if we are in characteristic $p$ where there is no logarithm. In characteristic 0, $f'/f = (\log f)'$.)

4. By QR in $\mathbb{F}_2[T]$,
   $$g_1^* \equiv g_2^* \mod \frac{1}{T^4} \implies [T^3, g_1] = [T^3, g_2].$$
   a) Compute $[T^3, T^{12} + T^4 + T^2 + T + 1]$. (The degree 12 polynomial is irreducible.)
   b) Prepare a table which indicates when $[T^3, \pi] = 0$ in terms of congruence conditions on $\pi^*$ in $\mathbb{F}_2[1/T]/(1/T)^4$.

   Continued fractions.

5. Compute the continued fraction expansion of $\sqrt{1 + 1/T}$ in $\mathbb{F}_2((1/T))$.

6. Let $F$ be a field with characteristic 2. We extend exercises 1 and 9 on Homework 2 to characteristic 2.
   a) For $f \in F((1/T))$, prove $f = \varphi(x) = x^2 + x$ for some $x \in F((1/T))$ if and only if the polynomial part of $f$ has the form $\varphi(g)$ for some polynomial $g \in F[T]$. (Hint: When $|y| < 1$, show $\sum_{n \geq 0} y^{2n}$ converges in $F((1/T))$ and has $\varphi$-value $y$. Also keep in mind, from Homework 2, that $[x^2] = [x]^2$ for any $x \in F((1/T))$ since $F$ has characteristic 2.)
   b) Determine which of the following rational functions has the form $\varphi(g)$ for some $g \in \mathbb{F}_2((1/T))$: $T/(T + 1)$, $(T^3 + T + 1)/(T + 1)$, $T^3/(T^2 + 1)$.
   c) For $b, c \in F[T]$ such that $b \notin F$, assume the equation $u^2 + buv + cv^2 = 1$ has a solution $u, v \in F[T]$ with $v \neq 0$. Prove $x^2 + bx + c$ has roots in $F((1/T))$ and $u/v$ is a convergent to the continued fraction of one of these roots. (Writing "$b \notin F"$ in the equivalent form "$b^2 - 4c \notin F,"$ as $2 = 0$, this result assumes a form identical to the case of characteristic $\neq 2$ on Homework 2.)

7. The polynomial $y^2 + Ty + T$ has a root in $\mathbb{F}_2((1/T))$ which begins $T + 1 + 1/T + 1/T^3 + \cdots$
   a) Find the complete Laurent expansion of this root, and also of the other root in $\mathbb{F}_2((1/T))$. 

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b) Find the (periodic) continued fraction expansions of these two roots.

c) If $u, v \in \mathbb{F}_2[T]$ is a solution to the Pell-type equation $u^2 + Tuv + Tv^2 = 1$ with $v \neq 0$, show that either $\deg u = \deg v$ or $\deg u = \deg v + 1$.

d) Use continued fractions to find a solution to $u^2 + Tuv + Tv^2 = 1$ in $\mathbb{F}_2[T]$ with $\deg u = \deg v = 2$, and then with $\deg u = 5, \deg v = 4$. In each case, which root of $y^2 + Ty + T$ has $u/v$ as a convergent?

8. Determine a nontrivial solution in $\mathbb{F}_2[T]$ to each of the following equations, or prove no nontrivial solution exists:

$$u^2 + (T + 1)uv + T^3v^2 = 1, \quad u^2 + Tuv + (T^2 + T)v^2 = 1, \quad u^2 + Tuv + (T^4 + T^3 + 1)v^2 = 1.$$  

For those equations without a nontrivial solution in $\mathbb{F}_2[T]$, can you find a nontrivial solution in $\mathbb{F}_4[T]$? (If you are not comfortable with $\mathbb{F}_4$, ignore this last part.)