ZETA AND L-FUNCTIONS

HOMEWORK 2
JULY 10, 2000

Due: Monday, July 17 at the beginning of class

Problems:
(1) Write the letter $\chi$ properly 25 times, by hand. (Warning: This is not the letter $x$! Note $\chi$ hangs partly below the line of writing, like $j$.)
(2) Use the Euler product for $\zeta(s)$ and $L(s)$ to prove
\[ \frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}, \quad \frac{1}{L(s)} = \sum_{n \geq 1} \frac{\mu(n)\chi_4(n)}{n^s} \]
for $s > 1$. Indicate which of the basic convergence theorems you use from the second technical handout.
(3) Let $a_1, a_2, \ldots$ be a totally multiplicative sequence (i.e., $a_{mn} = a_m a_n$ for all positive integers $m$ and $n$), with $|a_n| \leq 1$ for all $n$. Use appropriate convergence theorems from the second technical handout to prove that
\[ \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_p \frac{1}{1 - a_p p^{-s}} \]
for $s > 1$, where both product and sum can be rearranged.
(4) For an integer $n$, let
\[ \chi_3(n) = \begin{cases} 1, & \text{if } n \equiv 1 \mod 3, \\ -1, & \text{if } n \equiv 2 \mod 3, \\ 0, & \text{if } n \equiv 0 \mod 3. \end{cases} \]
Note $\chi_3(nn') = \chi_3(n)\chi_3(n')$ for all integers $n$ and $n'$. The series
\[ L(s, \chi_3) = \sum_{n \geq 1} \frac{\chi_3(n)}{n^s} \]
\[ = 1 - \frac{1}{2^s} - \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + \ldots \]
converges for $s > 0$, and for $s > 1$ we have
\[ L(s, \chi_3) = \prod_p \frac{1}{1 - \chi_3(p)p^{-s}}. \]
a) Prove by analytic methods that there are infinitely many primes $p \equiv 1 \mod 3$ and that there are infinitely many primes $p \equiv 2 \mod 3$. Mimic the argument used to handle the case of primes mod 4.
b) Give an elementary Euclid-style proof of this infinitude.
(5) For a polynomial \( f(T) \) in \( \mathbb{F}_3[T] \), set

\[
\chi(f) = \begin{cases} 
1, & \text{if } f(0) = 1, \\
-1, & \text{if } f(0) = 2, \\
0, & \text{if } f(0) = 0.
\end{cases}
\]

(Warning: Although \( 2 \equiv -1 \mod 3 \), it is not correct that \( \chi(f) = f(0) \); \( f(0) \) is in \( \mathbb{F}_3 \), while \( \chi(f) \) is in the real numbers.)

a) Show \( \chi(fg) = \chi(f)\chi(g) \) for all \( f, g \) in \( \mathbb{F}_3[T] \).

b) Define \( L(s, \chi) = \sum_{\text{monic } f} \chi(f)N_f^{-s} \).

Use appropriate convergence theorems to show this series converges when \( s > 1 \) and there is an Euler product

\[
L(s, \chi) = \prod_{\text{monic } \pi} \frac{1}{1 - \chi(\pi)N\pi^{-s}},
\]

where the order of addition and multiplication does not matter. (Note that unlike the case of \( L(s) \) from class, the \( L \)-function \( L(s, \chi) \) here is not initially defined for \( 0 < s \leq 1 \).)

c) The functions \( \chi_3 \) and \( \chi_4 \) on \( \mathbb{Z} \) are defined modulo 3 and 4, respectively. What should be the “modulus” for \( \chi \) on \( \mathbb{F}_3[T] \)?

d) For \( s > 1 \), prove \( L(s, \chi) = 1 \). (!!!)

e) Prove by analytic methods that in \( \mathbb{F}_3[T] \), there are infinitely many monic irreducibles with constant term 1 and infinitely many monic irreducibles with constant term 2.

(6) For a Gaussian integer \( \alpha = a + bi \), recall its norm is \( N \alpha = a^2 + b^2 \) and this norm is multiplicative by simple algebraic calculations.

a) For nonzero \( \alpha \) in \( \mathbb{Z}[i] \), prove \( \#\mathbb{Z}[i]/\alpha = N\alpha \). (This part is logically not needed for the remaining parts of the problem. Its purpose is to show the familiar norm function on \( \mathbb{Z}[i] \) can be thought of in the same combinatorial way as the norm on \( \mathbb{F}_p[T] \) and the absolute value on \( \mathbb{Z} \).

b) Define the zeta function of \( \mathbb{Z}[i] \) to be

\[
\zeta(\mathbb{Z}[i])(s) = \prod_{\text{(\pi)}} \frac{1}{1 - N\pi^{-s}},
\]

where \( \prod_{(\pi)} \) designates a product over nonassociate irreducibles of \( \mathbb{Z}[i] \). That is, out of every four associate irreducibles, one term is contributed to the product.

Prove the Euler product defining \( \zeta(\mathbb{Z}[i])(s) \) converges for \( s > 1 \) and then show

\[
\zeta(\mathbb{Z}[i])(s) = \sum_{(\alpha)} \frac{1}{N\alpha^s},
\]

where \( \sum_{(\alpha)} \) is a summation over nonassociate elements of \( \mathbb{Z}[i] \). Explicitly cite any convergence theorems you use for infinite series and products, as well as any arithmetic properties you use of \( \mathbb{Z}[i] \). (Hint: At most two nonassociate irreducible \( \pi \) divide any given integer prime \( p \), and \( N\pi \geq p \).)
c) For $s > 1$, use the Euler product definition of $\zeta_{\mathbb{Z}[i]}(s)$ to prove
\[
\log(\zeta_{\mathbb{Z}[i]}(s)) = \sum_{(n)} \frac{1}{N\pi^s} + g(s),
\]
where $g(s)$ is a Dirichlet series with terms $\geq 0$ which converges for $s > 1/2$.

d) For $s > 1$, prove $\zeta_{\mathbb{Z}[i]}(s) = \zeta(s)L(s)$, where $L(s)$ is the $L$-function of $\chi_4$ from lecture. This analytic identity encodes the description of how primes in $\mathbb{Z}$ factor in $\mathbb{Z}[i]$, and could be used to give an alternate proof of part c).

(7) You may be wondering why we did not define $\zeta_{\mathbb{Z}[i]}(s)$ as a sum or product over individual elements, rather than over classes of associate elements. To give a defining formula over individual elements in any kind of natural way, we want $\mathbb{Z}[i]$ to have some notion corresponding to “positive” in $\mathbb{Z}$ and “monic” in $\mathbb{F}_p[T]$. At the very least, we need a subset $H \subset \mathbb{Z}[i] - \{0\}$ with the following two properties:

i) Each nonzero Gaussian integer has a unique unit multiple in $H$.

ii) $H$ is closed under multiplication.

Note that we only describe $H$ by multiplicative properties, not by additive properties.

It is natural to include the following additional condition:

iii) $\mathbb{Z}^+ \subset H$.

Show a set $H$ of nonzero Gaussian integers satisfying properties i), ii), and iii) does not exist, and describe an example of a set $H$ satisfying i) and ii), but not iii).

Hint: Think about the associates of $1 + i$ and $(1 + i)^2$. 