Let $G$ be a locally compact abelian group and $\hat{G}$ be its dual group with the compact-open topology. To prove $\hat{G}$ is locally compact, the standard proof uses Banach algebras, Alaoglu’s theorem, $L^1$-$L^\infty$ duality, and a comparison between the compact-open topology and the topology of pointwise convergence on $\hat{G}$. We will give here a proof that the dual group is locally compact which uses only the compact-open topology on the dual group (no other topologies) and no Banach algebras, etc. Our main tool is the standard theorem describing when sets of functions are compact: Ascoli’s theorem.

**Lemma 1.** Fix $f \in L^1(G)$. For $y \in G$, let $L_yf : G \to \mathbb{C}$ by $(L_yf)(x) = f(yx)$. The map $y \mapsto L_yf$ from $G$ to $L^1(G)$ is continuous.

The lemma is proved by checking it directly on the dense subset $C_c(G) \subset L^1(G)$ and then extending it to all of $L^1(G)$ by an approximation argument. Details are left to the reader.

Next we prove a result about the decay of Fourier transforms. For $f \in L^1(G)$, its Fourier transform is $\hat{f}(\chi) = \int_G f(x)\chi(x) \, dx$, where $dx$ is some choice of Haar measure on $G$. Using the compact-open topology on $\hat{G}$, $\hat{f} : \hat{G} \to \mathbb{C}$ is (uniformly) continuous.

**Theorem 2.** If $f \in L^1(G)$ then $\hat{f} : \hat{G} \to \mathbb{C}$ “vanishes at $\infty$”: for any $\varepsilon > 0$ there is a compact set $C \subset \hat{G}$ such that $|\hat{f}(\chi)| < \varepsilon$ for $\chi \notin C$.

**Proof.** Since $\hat{f} : \hat{G} \to \mathbb{C}$ is continuous, our task is the same as showing for any $\varepsilon > 0$ that the (closed) set

$$\{\chi \in \hat{G} : |\hat{f}(\chi)| \geq \varepsilon\}$$

is compact in $\hat{G}$ using the compact-open topology.

Since $\hat{G}$ is a closed subset of the space $C(G,S^1)$ of continuous functions from $G$ to $S^1$, what we need to do is show the above set is compact in $C(G,S^1)$. For this, Ascoli’s theorem tells us exactly what has to be checked: equicontinuity of our set of characters at each point of $G$. Since we’re dealing with characters and the compact-open topology, it is enough to check equicontinuity of our set of characters at the identity $e$ of $G$. So for each $\delta > 0$ we want to find an open neighborhood $U = U_\delta$ of $e$ such that

$$y \in U, \quad |\hat{f}(\chi)| \geq \varepsilon \implies |\chi(y) - 1| < \delta.$$ 

It’s not evident how to turn a lower bound on the Fourier transform at $\chi$ into an upper bound on $\chi(y) - 1$. The trick is to get a bound on $|\chi(y) - 1|$ where $y$ doesn’t show up in $\chi(y)$ anymore.
For any $\chi \in \hat{G}$ such that $|\hat{f}(\chi)| \geq \varepsilon$ and any $y \in G$, we have

$$
\varepsilon|\chi(y) - 1| \leq |(\chi(y) - 1)\hat{f}(\chi)|
= \left| (\chi(y) - 1) \int_G f(x)\overline{\chi}(x) \, dx \right|
= \left| \int_G f(x)\overline{\chi(xy)} \, dx - \int_G f(x)\overline{\chi(x)} \, dx \right|
= \left| \int_G f(xy^{-1})\overline{\chi}(x) \, dx - \int_G f(x)\overline{\chi}(x) \, dx \right|
= \left| \int_G (f(xy^{-1}) - f(x))\overline{\chi}(x) \, dx \right|
\leq \int_G |f(xy^{-1}) - f(x)| \, dx
= |L_{y^{-1}}f - f|_1,
$$

so

$$
|\chi(y) - 1| \leq \frac{1}{\varepsilon}|L_{y^{-1}}f - f|_1.
$$

From continuity of $y \mapsto L_yf$ and continuity of inversion on $G$, $|L_{y^{-1}}f - f|_1 \to 0$ as $y \to e$ in $G$. Therefore $|\chi(y) - 1| < \delta$ for all $y$ near $e$, and that level of nearness to $e$ gives us the desired set $U$. $\square$

**Remark 3.** Theorem 2 is a generalization of the Riemann–Lebesgue lemma, which is the special case $G = S^1 = \mathbb{R}/2\pi\mathbb{Z}$: if $f: \mathbb{R} \to \mathbb{C}$ is $2\pi$-periodic and integrable then its Fourier coefficients $\hat{f}(n) = \int_0^{2\pi} f(x)e^{-2\piinx} \, dx$ tend to 0 as $|n| \to \infty$.

**Corollary 4.** When $G$ is locally compact, so is $\hat{G}$ in the compact-open topology: if $0 < \varepsilon < 1$ and $K \subset G$ is compact then the neighborhood $N(K, \varepsilon) = \{\chi \in \hat{G} : |\chi(x) - 1| < \varepsilon \text{ for all } x \in K\}$ of the trivial character has compact closure in $\hat{G}$.

**Proof.** We copy the argument from Hewitt & Ross vol. 1, page 362.

Let $f = \xi_K$ be the characteristic function of $K$, so $f \in L^1(G)$. For any $\chi \in \hat{G}$,

$$
\mu(K) = \int_G \xi_K \, dx = \int_G \xi_K \cdot (1 - \chi) \, dx + \int_G \xi_K \cdot \chi \, dx.
$$

Taking absolute values, if $\chi \in N(K, \varepsilon)$ then $\mu(K) \leq \mu(K)\varepsilon + |\xi_K(\chi)|$, so

$$
|\xi_K(\chi)| \geq (1 - \varepsilon)\mu(K).
$$

By Theorem 2, the set of $\chi$ fitting this inequality is a compact set, so $N(K, \varepsilon)$ has compact closure in $\hat{G}$. $\square$