EULER’S CONSTANT

PAT SMITH

Math 2784 (or 2794W)
University of Connecticut

Date: Oct. 2, 2014.
1. Introduction

One of the proofs of the divergence of the harmonic series
\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots \]
is based on comparing the partial sum
\[ (1.1) \quad 1 + \frac{1}{2} + \cdots + \frac{1}{n} \]
to the integral \( \int_1^n \frac{1}{t} \, dt \). That they might be comparable is suggested by the integral test, which says an infinite series \( \sum_{k=1}^{\infty} f(k) \) converges or diverges (under reasonable conditions) in the same way as the integral \( \int_1^{\infty} f(t) \, dt \). Thus we can expect the sum (1.1) is approximately as large as \( \int_1^n \frac{1}{t} \, dt = \log n \), which diverges as \( n \to \infty \). Euler discovered in 1740 that in fact the difference between (1.1) and \( \log n \), namely
\[ (1.2) \quad 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n, \]
doesn’t go to 0 but does have a limit as \( n \to \infty \). The limit is called Euler’s constant, is denoted \( \gamma \), and
\[ \gamma \approx 0.577215664901532860 \ldots \]
(Euler miscomputed the 16th digit as 5 instead of 8. See [1, Fig. 10.2].) So for large \( n \),
\[ 1 + 1/2 + \cdots + 1/n \approx \log n + \gamma. \]

We will show the limit of (1.2) exists and give an error estimate so that we can prove rigorously that the decimal expansion of \( \gamma \) starts out as .57.

2. Existence

**Theorem 2.1 (Euler).** The limit
\[ \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right) \]
exists.

*Proof.* First let’s look at some numerical data, to understand the approach we will take. In Table 1 we list the differences
\[ a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \]
for small \( n \) and see how these numbers behave: they are positive and decreasing.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 1 + 1/2 + \cdots + 1/n - \log n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>.806852</td>
</tr>
<tr>
<td>3</td>
<td>.734721</td>
</tr>
<tr>
<td>4</td>
<td>.6967038</td>
</tr>
</tbody>
</table>

**Table 1**

Let’s show the trend from the table really persists: for all \( n \geq 1 \), \( a_n > 0 \) and \( a_n > a_{n+1} \).
The positivity of each \( a_n \) can be checked by writing \( \log n \) as an integral from 1 to \( n \) and breaking up the integral into integrals over \([k, k+1]\) for \( 1 \leq k \leq n - 1 \):

\[
\begin{align*}
a_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \\
&= \sum_{k=1}^{n} \frac{1}{k} - \int_{1}^{n} \frac{dt}{t} \\
&= \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n} - \sum_{k=1}^{n-1} \int_{k}^{k+1} \frac{dt}{t} \\
&= \sum_{k=1}^{n-1} \left( \frac{1}{k} - \int_{k}^{k+1} \frac{dt}{t} \right) + \frac{1}{n}.
\end{align*}
\]

(2.1)

For \( k \leq t \leq k+1 \), \( 1/t \leq 1/k \), so

\[
\int_{k}^{k+1} \frac{dt}{t} \leq \int_{k}^{k+1} \frac{dt}{k} = \frac{1}{k} \int_{k}^{k+1} \frac{dt}{k} = \frac{1}{k^2}.
\]

Therefore the terms in the sum in (2.1) are all nonnegative, so \( a_n \geq 1/n > 0 \).

To show \( a_n > a_{n+1} \), let’s look at the difference

\[
a_n - a_{n+1} = \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right) - \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} - \log(n+1) \right)
\]

\[
= \log(n+1) - \log n - \frac{1}{n+1}
\]

\[
= \log \left( 1 + \frac{1}{n} \right) - \frac{1}{n+1}.
\]

Recall \( \log(1+x) = x - x^2/2 + x^3/3 - \cdots \) for \(-1 < x \leq 1\). We care about \( x = 1/n \). The series for \( \log(1+x) \) is alternating, so for \( x > 0 \) we have \( \log(1+x) > x - x^2/2 \). Setting \( x = 1/n \), this inequality becomes

\[
\log \left( 1 + \frac{1}{n} \right) > \frac{1}{n} - \frac{1}{2} \left( \frac{1}{n} \right)^2 = \frac{2n-1}{2n^2}
\]

for \( n \geq 1 \). Therefore

\[
a_n - a_{n+1} > \frac{2n-1}{2n^2} - \frac{1}{n+1} = \left( \frac{2n-1}{2n^2} - \frac{1}{2n^2(n+1)} \right) = \frac{n-1}{2n^2(n+1)} \geq 0,
\]

so \( a_n > a_{n+1} \). \( \square \)

A fundamental property of the real numbers is that any sequence that is monotonic (i.e., increasing or decreasing) and bounded has a limit. Therefore since the sequence \( \{a_n\} \) is decreasing and positive, it has a limit. That proves \( \gamma = \lim_{n \to \infty} a_n \) exists.

### 3. Estimating Euler’s constant

To estimate \( \gamma \), we will establish a bound on how far each \( a_n \) is from \( \gamma \) in terms of \( n \).

**Corollary 3.1.** For any \( n \geq 1 \),

\[
0 < a_n - \gamma \leq \frac{1}{n}.
\]
Proof. We will use the formula
\[ a_n - a_{n+1} = \log \left( 1 + \frac{1}{n} \right) - \frac{1}{n+1} \]
from the proof of Theorem 2.1. Previously we plugged into this a lower bound on \( \log(1+1/n) \). Now we plug in an upper bound: for \( 0 < x < 1 \), the series for \( \log(1 + x) \) tells us that \( \log(1 + x) < x \), so in particular \( \log(1 + 1/n) < 1/n \). This tells us
\[ a_n - a_{n+1} < \frac{1}{n} - \frac{1}{n+1}, \tag{3.1} \]
Pick a positive integer \( N \). For any \( M > N \), we have \( a_M < a_N \) since the sequence \( \{a_n\} \) is decreasing, and therefore
\[
0 < a_N - a_M = (a_N - a_{N+1}) + (a_{N+1} - a_{N+2}) + \cdots + (a_{M-1} - a_M) < \left( \frac{1}{N} - \frac{1}{N+1} \right) + \left( \frac{1}{N+1} - \frac{1}{N+2} \right) + \cdots + \left( \frac{1}{M-1} - \frac{1}{M} \right) = \frac{1}{N} - \frac{1}{M},
\]
where the last equation comes from the telescoping sum before it (adjacent terms cancel). Letting \( M \to \infty \),
\[
0 < a_N - \gamma \leq \frac{1}{N},
\]
Since \( N \) was arbitrary, we are done. \( \square \)

Example 3.2. If we want to estimate \( \gamma \) to 2 decimal places, let’s compute \( a_{1000} \) since Corollary 3.1 tells us that
\[
0 < a_{1000} - \gamma \leq \frac{1}{1000},
\]
so
\[ a_{1000} - \frac{1}{1000} < \gamma < a_{1000}. \tag{3.2} \]
From a computer, \( a_{1000} = .577715 \ldots \). Therefore \( a_{1000} - 1/1000 = .576715 \ldots \), so (3.2) implies
\[ .576 < \gamma < .5778, \]
which means \( \gamma \) is .57 to two decimal places.

Another proof of Theorem 2.1, which incorporates Corollary 3.1 directly into it, can be found in [2, p. 268].

While the more widely known constants \( \pi \) and \( e \) are known to be irrational, it is believed that \( \gamma \) is irrational but nobody has proved this and nobody has any compelling approaches to settle the question. The numbers \( \pi \) and \( e \) are known to be transcendental (that is, they are not the root of any nonconstant polynomial with rational coefficients, so in a sense they’re more irrational than \( \sqrt{2} \), which is a root of \( x^2 - 2 \)), and it is expected that \( \gamma \) is transcendental also, but that would lie much deeper than irrationality.
REFERENCES