1. Introduction

Set
\[ G = \left\{ \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} : x > 0, y \in \mathbb{R} \right\}, \]
which is a group under matrix multiplication:
\[ \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} u & v \\ 0 & 1/u \end{pmatrix} = \begin{pmatrix} xu & xv + y/u \\ 0 & 1/xu \end{pmatrix}, \quad \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & -y \\ 0 & x \end{pmatrix}. \]

We geometrically represent \( \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \) as the point \((x, y)\) in the plane. So \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) corresponds to \((1, 0)\) and we plot \( g = \begin{pmatrix} 2 & 2 \\ 0 & 1/2 \end{pmatrix} \), \( h = \begin{pmatrix} 3 & 1 \\ 0 & 1/3 \end{pmatrix} \), and several powers and products in Figure 1. Note \( gh \neq hg \).
In $G$, there are two “natural” subgroups

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} : x > 0 \right\}, \quad K = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\}.$$  

They are pictured below in Figure 2 as the points $(x,0)$ for $H$ and the points $(1,y)$ for $K$.

![Figure 2. The subgroups $H$ and $K$.](image)

In Section 2 we will make pictures of conjugacy classes and conjugate subgroups, and in Section 3 we will see pictures of the left and right cosets of $H$ and $K$.

## 2. Conjugacy Classes and Conjugate Subgroups

The conjugate of $\begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}$ by $\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$ is

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1/b \end{pmatrix}^{-1} = \begin{pmatrix} x & ab(1/x - x) + a^2y \\ 0 & 1/x \end{pmatrix}. \quad (2.1)$$

Equation (2.1) tells us **conjugate elements of $G$ have the same same upper left entry**. Therefore in our picture of $G$, conjugate elements of $G$ have the same first coordinate: they must lie on the same vertical line. We can use the formula (2.1) to compute a conjugacy class: fix $x$ and $y$, and let $a$ and $b$ vary on the right side of (2.1). Here are the results.

- The conjugacy class of the identity $(1,0)$ is itself. See the green dot in Figure 3.
- Conjugates of $(\frac{1}{0} \frac{1}{1})$ are found by setting $x = y = 1$ on the right side of (2.1). We get $(\frac{1}{0} a^2)$ for all $a > 0$, which in Figure 3 is the red half-line through $(1,1)$ above the $x$-axis.
• Conjugates of \((\frac{1}{0} - \frac{1}{1})\) are \((\frac{1}{0} - \frac{a^2}{1})\) for all \(a > 0\), which in Figure 3 is the blue half-line through \((1, -1)\) below the \(x\)-axis.

• We now determine the conjugacy class of \((\frac{x}{0} \frac{0}{1/x})\), where \(x \neq 1\). A matrix conjugate to this has the form \((\frac{x}{0} \frac{y}{1/x})\) for some \(y\). We will now show, when \(x \neq 1\), that any matrix \((\frac{x}{0} \frac{y}{1/x})\) is conjugate to \((\frac{x}{0} \frac{0}{1/x})\). This would mean that in Figure 3, the conjugacy class of \((\frac{x}{0} \frac{0}{1/x})\) is represented by the whole vertical line through \((x, 0)\).

To prove our description of the conjugacy class of \((\frac{x}{0} \frac{0}{1/x})\) is correct, this conjugacy class includes the matrices \((\frac{1}{0} \frac{b}{1})(\frac{x}{0} \frac{0}{1/x})(\frac{1}{0} \frac{b}{1})^{-1} = (\frac{x}{0} \frac{b(x-1/x)}{1/x})\), with \(b\) running through all real numbers. Here \(b\) is variable and \(x\) is fixed. Since \(x \neq 1\) we have \(x - 1/x \neq 0\), so the upper right entry of the conjugate matrix runs through all real numbers as \(b\) varies. See the orange and purple vertical lines in Figure 3 corresponding to \(x = 3\) and \(x = 5\).

\[
f(1,1)
\]
\[
(1,0)
\]
\[
(1,-1)
\]
\[
(3,0)
\]
\[
(5,0)
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Conjugacy classes of \((\frac{1}{0} \frac{1}{1}), (\frac{1}{0} \frac{-1}{1}), (\frac{1}{0} \frac{0}{1}), (\frac{3}{0} \frac{0}{1/3}),\) and \((\frac{5}{0} \frac{0}{1/5})\).}
\end{figure}

Turning from conjugacy classes of elements to conjugate subgroups, we will compute the subgroups conjugate to \(H = \{(\frac{x}{0} \frac{0}{1/x}) : x > 0\}\) and to \(K = \{(\frac{1}{0} \frac{y}{1}) : y \in \mathbb{R}\}\). The answers in these two cases will be very different.

For \(a > 0\) and \(b \in \mathbb{R}\), we have by equation (2.1) with \(y = 0\) that \((\frac{a}{0} \frac{b}{1/a})(\frac{x}{0} \frac{0}{1/x})(\frac{a}{0} \frac{b}{1/a})^{-1} = (\frac{x}{0} \frac{ab(1/x-x)}{1/x})\), so the subgroup conjugate to \(H\) by \((\frac{a}{0} \frac{b}{1/a})\) is

\[
(2.2) \quad \left(\begin{array}{cc}
\frac{a}{0} & \frac{b}{1/a} \\
\frac{0}{1/a} & \frac{1}{1/a}
\end{array}\right) H \left(\begin{array}{cc}
\frac{a}{0} & \frac{b}{1/a} \\
\frac{0}{1/a} & \frac{1}{1/a}
\end{array}\right)^{-1} = \left\{\left(\begin{array}{cc}
x & \frac{ab(1/x-x)}{1/x} \\
0 & \frac{1}{1/x}
\end{array}\right) : x > 0\right\}.
\]
On the right side of (2.2), $a$ and $b$ are fixed and $x$ varies. Since $a$ and $b$ occur on the right side of (2.2) only in the context of $ab$, for nonzero $b$ we have

$$
\left( \begin{array}{cc} a & b \\ 0 & 1/a \end{array} \right) H \left( \begin{array}{cc} a & b \\ 0 & 1/a \end{array} \right)^{-1} = \left( \begin{array}{cc} ab & 1 \\ 0 & 1/ab \end{array} \right) H \left( \begin{array}{cc} ab & 1 \\ 0 & 1/ab \end{array} \right)^{-1}.
$$

So conjugating $H$ by an element of $G$ that is not in $H$ (meaning $b \neq 0$ on the left) has the same
effect as conjugating $H$ by some matrix of the form $(t \ 1/t)$ with $1$ in the upper right.

As an example,

$$
\left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) H \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)^{-1} = \left\{ \left( \begin{array}{cc} x & 1/x-x \\ 0 & 1/x \end{array} \right) : x > 0 \right\}.
$$

In Figure 4 this conjugate subgroup is represented by the set of all points $(x, 1/x - x)$ with $x > 0$, which is the graph of $y = 1/x - x$ for $x > 0$ (in red). The conjugate subgroup $(2 \ 1/2) H (2 \ 1/2)^{-1}$ is the set of matrices $(x \ 2(1/x-x) \\ 0 \ 1/x)$, which in Figure 4 is represented by the graph of $y = 2(1/x - x)$ for $x > 0$ (in green). More generally, from (2.2) the subgroup conjugate to $H$ by $(a \ 1/a)$ is represented as the graph of $y = a(1/x - x)$ for $x > 0$. As $a$ varies, these curves are pictured for different subgroups conjugate to $H$.

![Figure 4](image-url)

**Figure 4.** Conjugating $H$ by $(1/4 \ 1)\), $(−1/4 \ 1)$, $(2 \ 1/2)$, $(−2 \ 1/2)$, $(1/4 \ 1)$, and $(−1/4 \ 1)\).
What are subgroups conjugate to $K$? Since \( (\begin{array}{cc} a & b \\ 0 & 1/a \end{array}) (\begin{array}{cc} 1 & y \\ 0 & 1 \end{array}) (\begin{array}{cc} a & b \\ 0 & 1/a \end{array})^{-1} = (\begin{array}{cc} 1 & a^2y \\ 0 & 1 \end{array}) \) we get

\[
(\begin{array}{cc} a & b \\ 0 & 1/a \end{array}) K (\begin{array}{cc} a & b \\ 0 & 1/a \end{array})^{-1} = \left\{ (\begin{array}{cc} 1 & a^2y \\ 0 & 1 \end{array}) : y \in \mathbb{R} \right\} = \left\{ (\begin{array}{cc} 1 & t \\ 0 & 1 \end{array}) : t \in \mathbb{R} \right\} = K.
\]

That is, the **only** subgroup of $G$ that is conjugate to $K$ is $K$. See Figure 5. This is an important difference between $H$ and $K$.

\[ K = gKg^{-1} \]

\[ (1,0) \]

\[ x \]

\[ y \]

**Figure 5.** The only conjugate subgroup of $K$ is $K$.

### 3. Cosets

We will draw pictures for the left and right cosets of the subgroups $H$ and $K$. For $g = (\begin{array}{cc} a & b \\ 0 & 1/a \end{array})$, a typical element in $gH$ is

\[
(\begin{array}{cc} a & b \\ 0 & 1/a \end{array}) (\begin{array}{cc} x & 0 \\ 0 & 1/x \end{array}) = (\begin{array}{cc} ax & b/x \\ 0 & 1/ax \end{array})
\]

where $x > 0$. Letting $x$ run over all positive numbers, by a change of variables

\[
gH = \left\{ (\begin{array}{cc} ax & b/x \\ 0 & 1/ax \end{array}) : x > 0 \right\} = \left\{ (\begin{array}{cc} t & ab/t \\ 0 & 1/t \end{array}) : t > 0 \right\},
\]

which is pictured as the graph of $y = ab/x$ for $x > 0$: the **branch of a hyperbola** passing through $(a, b)$. See Figure 6. The left $H$-cosets are branches of hyperbolas, which fill up $G$ without overlapping.
A typical element in the right coset $Hg$ is
$$ \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} ax & bx \\ 0 & 1/ax \end{pmatrix} $$
for $x > 0$. Letting $x$ run over all positive numbers,
$$ Hg = \left\{ \begin{pmatrix} ax & bx \\ 0 & 1/ax \end{pmatrix} : x > 0 \right\} = \left\{ \begin{pmatrix} t & (b/a)t \\ 0 & 1/t \end{pmatrix} : t > 0 \right\}, $$
which is pictured as the graph of the ray $y = (b/a)x$ coming out of the origin and passing through $(a,b)$. See Figure 7. The left and right $H$-cosets look quite different, but in each case the cosets on the same side (all left or all right) fill up $G$ without overlapping.

Turning to the left and right cosets of $K$, a typical element in $gK$ is
$$ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ay + b \\ 0 & 1/a \end{pmatrix}. $$
As $y$ runs over all real numbers, $ay + b$ runs over all real numbers, so
$$ gK = \left\{ \begin{pmatrix} a & y \\ 0 & 1/a \end{pmatrix} : y \in \mathbb{R} \right\}, $$
which is pictured as the vertical line $x = a$. Similarly, a typical element of the right coset $Kg$ is
$$ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} a & b + y/a \\ 0 & 1/a \end{pmatrix}, $$
and as $y$ runs over $\mathbb{R}$ the numbers $b + y/a$ run over $\mathbb{R}$, so $Kg = gK$ for every $g \in G$. The left $K$-cosets and the right $K$-cosets are both the collection of all vertical lines, which fill up $G$ without overlaps. See Figure 8.