A 2-PARAMETER NONABELIAN GROUP

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1. Introduction

Set

\[ G = \left\{ \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} : x > 0, y \in \mathbb{R} \right\}, \]

which is a group under matrix multiplication:

\[
\begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} u & v \\ 0 & 1/u \end{pmatrix} = \begin{pmatrix} xu & xv + y/u \\ 0 & 1/xu \end{pmatrix}, \quad \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & -y \\ 0 & x \end{pmatrix}.
\]

We geometrically represent \( \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \) as the point \((x, y)\) in the plane. So \(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) corresponds to \((1, 0)\) and we plot \(g = \begin{pmatrix} 2 & 2 \\ 0 & 1/2 \end{pmatrix}\), \(h = \begin{pmatrix} 3 & 1 \\ 0 & 1/3 \end{pmatrix}\), and several powers and products in Figure 1. Note \(gh \neq hg\).

![Figure 1. Powers and products of \(g = \begin{pmatrix} 2 & 2 \\ 0 & 1/2 \end{pmatrix}\) and \(h = \begin{pmatrix} 3 & 1 \\ 0 & 1/3 \end{pmatrix}\) in \(G\).](image-url)
In $G$, there are two “natural” subgroups

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} : x > 0 \right\}, \quad K = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\}. $$

They are pictured below in Figure 2 as the points $(x, 0)$ for $H$ and the points $(1, y)$ for $K$.

![Figure 2. The subgroups $H$ and $K$.]

In Section 2 we will make pictures of conjugacy classes and conjugate subgroups, and in Section 3 we will see pictures of the left and right cosets of $H$ and $K$.

2. CONJUGACY CLASSES AND CONJUGATE SUBGROUPS

The conjugate of $\begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}$ by $\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$ is

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1/b \end{pmatrix}^{-1} = \begin{pmatrix} x & ab(1/x - x) + a^2 y \\ 0 & 1/x \end{pmatrix}.$$  

Equation (2.1) tells us **conjugate elements of $G$ have the same same upper left entry**. Therefore in our picture of $G$, conjugate elements of $G$ have the same first coordinate: they must lie on the same vertical line. We can use the formula (2.1) to compute a conjugacy class: fix $x$ and $y$, and let $a$ and $b$ vary on the right side of (2.1). Here are the results.

- The conjugacy class of the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is itself. See the green dot in Figure 3.
- Conjugates of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are found by setting $x = y = 1$ on the right side of (2.1). We get $\begin{pmatrix} 1 & a^2 \\ 0 & 1 \end{pmatrix}$ for all $a > 0$, which in Figure 3 is the red half-line through $(1, 1)$ above the $x$-axis.
- Conjugates of $\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ are $\begin{pmatrix} 1 & -a^2 \\ 0 & -1 \end{pmatrix}$ for all $a > 0$, which in Figure 3 is the blue half-line through $(1, -1)$ below the $x$-axis.
- We now determine the conjugacy class of $\begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}$, where $x > 0$ and $x \neq 1$. A conjugate matrix has the form $\begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}$ for some $y$. We will now show, for $x > 0$ and $x \neq 1$, that the
A 2-PARAMETER NONABELIAN GROUP

matrix \( \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \) for all \( y \in \mathbb{R} \) is conjugate to \( \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \). This would mean that in Figure 3, the conjugacy class of \( \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \) for \( x > 0 \) with \( x \neq 1 \) is represented by the whole vertical line through \( (x, 0) \).

To prove our description of the conjugacy class of \( \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \) is correct, this conjugacy class includes the matrices
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x & b(x-1/x) \\ 0 & 1/x \end{pmatrix},
\]
with \( b \) running through all real numbers. Here \( b \) is variable and \( x \) is fixed. Since \( x > 0 \) and \( x \neq 1 \) we have \( x - 1/x \neq 0 \), so the upper right entry of the conjugate matrix runs through all real numbers as \( b \) varies.

See the orange and purple vertical lines in Figure 3 corresponding to \( x = 3 \) and \( x = 5 \).

\[\text{Figure 3. Conjugacy classes of } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \text{ and } \begin{pmatrix} 5 & 0 \\ 0 & 1/5 \end{pmatrix}.\]

Turning from conjugacy classes of elements to conjugate subgroups, we will compute the subgroups of \( G \) that are conjugate to \( H = \{ (x, 0) : x > 0 \} \) and to \( K = \{ (y, 0) : y \in \mathbb{R} \} \). The answers in these two cases will be very different.

For \( a > 0 \) and \( b \in \mathbb{R} \), we have by equation (2.1) with \( y = 0 \) that
\[
\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} (x, 0, 0, 1/x) \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}^{-1} = \begin{pmatrix} x & ab(1/x-x) \\ 0 & 1/x \end{pmatrix},
\]
so the subgroup conjugate to \( H \) by \( \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \) is
\[
(2.2) \quad \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} H \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}^{-1} = \left\{ \begin{pmatrix} x & ab(1/x-x) \\ 0 & 1/x \end{pmatrix} : x > 0 \right\}.
\]

On the right side of (2.2), \( a \) and \( b \) are fixed and \( x \) varies. Since \( a \) and \( b \) occur on the right side of (2.2) only through \( ab \), conjugating \( H \) by matrices in \( G \) whose top two entries have the same product leads to the same conjugate subgroup to \( H \). Thus for \( b > 0 \)
\[
\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} H \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}^{-1} = \begin{pmatrix} ab & 1 \\ 0 & 1/ab \end{pmatrix} H \begin{pmatrix} ab & 1 \\ 0 & 1/ab \end{pmatrix}^{-1}
\]
since $a \cdot b = ab \cdot 1$, and for $b < 0$

$$
\begin{pmatrix}
a & b \\
0 & 1/a
\end{pmatrix} H \begin{pmatrix}
a & b \\
0 & 1/a
\end{pmatrix}^{-1} = \begin{pmatrix} a|b| & -1 \\
0 & 1/a|b|
\end{pmatrix} H \begin{pmatrix} a|b| & -1 \\
0 & 1/a|b|
\end{pmatrix}^{-1}
$$

since $a \cdot b = a|b| \cdot (-1)$. Thus conjugating $H$ by an element of $G$ that is not in $H$ (meaning $b \neq 0$)

has the same effect as conjugating $H$ by a matrix of the form $(t \ 1/t \ 0 \ 1)$ or $(t^{-1} \ 0 \ 1/t \ 0)$, where $t > 0$.

As an example,

$$
\begin{pmatrix} 1 & 1 \\
0 & 1
\end{pmatrix} H \begin{pmatrix} 1 & 1 \\
0 & 1
\end{pmatrix}^{-1} = \left\{ \begin{pmatrix} x & 1/x - x \\
0 & 1/x
\end{pmatrix} : x > 0 \right\}.
$$

Figure 4. Conjugating $H$ by $(1 \ 1 \ 0 \ 1)$, $(1^{-1} \ 0 \ 1)$, $(2 \ 1/2 \ 0 \ 1)$, $(2^{-1} \ 0 \ 1/2 \ 0)$, $(1/4 \ 1 \ 0 \ 4)$, and $(1/4^{-1} \ 0 \ 4)$.

In Figure 4 this conjugate subgroup is represented by the set of all $(x, 1/x - x)$ with $x > 0$, which is the graph of $y = 1/x - x$ for $x > 0$ (in red). The conjugate subgroup $(2 \ 1/2 \ 0 \ 1)^{-1} H (2 \ 1/2 \ 0 \ 1)$ is all $(x \ 2(1/x - x))$, which in Figure 4 is represented by the graph of $y = 2(1/x - x)$ for $x > 0$ (in green).

More generally, from (2.2) the subgroup conjugate to $H$ by $(a \ 1/a)$ is represented as the graph of $y = a(1/x - x)$ for $x > 0$ and the subgroup conjugate to $H$ by $(a^{-1} \ 0 \ 1/a)$ is represented as the graph of $y = -a(1/x - x)$ for $x > 0$. These curves are pictured in Figure 4 for different $a$.

What subgroups in $G$ are conjugate to $K$? Since $(a \ b \ 0 \ 1)(\ 1/y \ 0 \ 1)^{-1} = (1 \ a^2 y \ 0 \ 1)$ we get

$$
\begin{pmatrix} a & b \\
0 & 1/a
\end{pmatrix} K \begin{pmatrix} a & b \\
0 & 1/a
\end{pmatrix}^{-1} = \left\{ \begin{pmatrix} 1 \ a^2 y \ 1 \\
0 \ 1
\end{pmatrix} : y \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \ t \\
0 \ 1
\end{pmatrix} : t \in \mathbb{R} \right\} = K,
$$
so the only subgroup of $G$ conjugate to $K$ is $K$. See Figure 5.

![Figure 5. The only conjugate subgroup of $K$ is $K$.](image)

3. Cosets

We will draw pictures for the left and right cosets of the subgroups $H$ and $K$. For $g = \left( \begin{array}{cc} a & b \\ 0 & 1/a \end{array} \right)$, a typical element in $gH$ is

$$
\left( \begin{array}{cc} a & b \\ 0 & 1/a \end{array} \right) \left( \begin{array}{cc} x & 0 \\ 0 & 1/x \end{array} \right) = \left( \begin{array}{cc} ax & b/x \\ 0 & 1/ax \end{array} \right)
$$

where $x > 0$. Letting $x$ run over all positive numbers, by a change of variables

$$
gH = \left\{ \left( \begin{array}{cc} ax & b/x \\ 0 & 1/ax \end{array} \right) : x > 0 \right\} = \left\{ \left( \begin{array}{cc} t & ab/t \\ 0 & 1/t \end{array} \right) : t > 0 \right\}
$$

which is pictured in Figure 6 as the graph of $y = ab/x$ for $x > 0$: the branch of a hyperbola passing through $(a, b)$. The left $H$-cosets are branches of hyperbolas that fill up $G$ without overlapping.

A typical element in the right coset $Hg$ is

$$
\left( \begin{array}{cc} x & 0 \\ 0 & 1/x \end{array} \right) \left( \begin{array}{cc} a & b \\ 0 & 1/a \end{array} \right) = \left( \begin{array}{cc} ax & bx \\ 0 & 1/ax \end{array} \right)
$$
Figure 6. The left cosets of $H$: hyperbolas $xy = \text{constant}$, $x > 0$.

for $x > 0$. Letting $x$ run over all positive numbers,

$$Hg = \left\{ \begin{pmatrix} ax & bx \\ 0 & 1/ax \end{pmatrix} : x > 0 \right\} = \left\{ \begin{pmatrix} t & (b/a)t \\ 0 & 1/t \end{pmatrix} : t > 0 \right\},$$

which is pictured in Figure 7 as the graph of the ray $y = (b/a)x$ coming out of the origin and passing through $(a, b)$. The right $H$-cosets are rays that fill up $G$ without overlapping.

Turning to the left and right cosets of $K$, a typical element in $gK$ is

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ay + b \\ 0 & 1/a \end{pmatrix}.$$  

As $y$ runs over all real numbers, $ay + b$ runs over all real numbers, so

$$gK = \left\{ \begin{pmatrix} a & y \\ 0 & 1/a \end{pmatrix} : y \in \mathbb{R} \right\},$$

which is pictured as the vertical line $x = a$. Similarly, a typical element of the right coset $Kg$ is

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} a & b + y/a \\ 0 & 1/a \end{pmatrix},$$

and as $y$ runs over $\mathbb{R}$ the numbers $b + y/a$ run over $\mathbb{R}$, so $Kg = gK$ for each $g \in G$. The left $K$-cosets and right $K$-cosets are each the collection of all vertical lines, which fill up $G$ without overlaps. See Figure 8.
Figure 7. The right cosets of $H$: rays coming out of $(0,0)$.

Figure 8. The left and right cosets of $K$: vertical lines.