Let $K$ be a field. We want to construct an algebraic closure of $K$, i.e., an algebraic extension of $K$ which is algebraically closed. It will be built as the quotient of a polynomial ring in a very large number of variables.

For each nonconstant monic polynomial $f(X)$ in $K[X]$, let its degree be $n_f$ and let $t_{f,1}, \ldots, t_{f,n_f}$ be independent variables. Let $A = K[\{t_{f,i}\}]$ be the polynomial ring generated over $K$ by independent variables doubly indexed by every nonconstant monic $f \in K[X]$ and $1 \leq i \leq n_f$. This is a very large polynomial ring containing $K$.

Let $I$ be the ideal in $A$ generated by the coefficients of all the difference polynomials $f(X) - \prod_{i=1}^{n_f} (X - t_{f,i}) \in A[X]$ as $f$ runs over nonconstant monic polynomials in $K[X]$. Working modulo $I$ we have $f(X) \equiv \prod_{i=1}^{n_f} (X - t_{f,i})$, so $f(X)$ splits into linear factors in $(A/I)[X]$. We want to use a maximal ideal in place of $I$ since working modulo a maximal ideal would give a complete splitting of every $f(X)$ from $K[X]$ over a field.

**Lemma 1.** The ideal $I$ is proper: $1 \notin I$.

**Proof.** This will follow from the existence of a splitting field of any nonconstant polynomial in $K[X]$.

Suppose $1 \in I$. Then we can write $1$ as a finite sum $\sum_{j=1}^{m} a_j c_j$, where $c_j \in I$ and $a_j \in A$. Each $c_j$ is a coefficient in some difference

$$f_j(X) - \prod_{i=1}^{n_j} (X - t_{f_j,i}),$$

where $f_j(X)$ is monic in $K[X]$ and $n_j = \deg(f_j)$. There is a (finite) field extension $L/K$ in which the finitely many $f_j$'s all split completely, say $f_j(X) = \prod_{i=1}^{n_j} (X - r_{f_j,i})$ in $L[X]$. (Some numbers in the list $r_{f_j,1}, \ldots, r_{f_j,n_j}$ might be repeated.) We can use the roots $r_{f_j,i}$ of $f_1(X), \ldots, f_m(X)$ to construct a ring homomorphism $\varphi$ from $A = K[\{t_{f,j}\}]$ to $L$ by substitution: $\varphi$ fixes $K$, $\varphi(t_{f_j,i}) = r_{f_j,i}$ for $1 \leq i \leq n_j$, and $\varphi(t_{f_j,i}) = 0$ if $f$ is not one of the $f_j$'s. Extend $\varphi$ to a homomorphism $A[X] \to L[X]$ by acting on coefficients. The polynomial in (1) is mapped by $\varphi$ to

$$f_j(X) - \prod_{i=1}^{n_j} (X - r_{f_j,i}) = 0 \text{ in } L[X],$$

so every coefficient in (1) is mapped by $\varphi$ to $0$ in $L$. In particular, $\varphi(c_j) = 0$. Thus $\varphi$ sends the equation $1 = \sum_{j=1}^{m} a_j c_j$ in $A$ to the equation $1 = 0$ in $L$, and that is a contradiction. □

Since $I$ is a proper ideal, Zorn’s lemma guarantees that $I$ is contained in some maximal ideal $\mathfrak{m}$ in $A$. (I suspect $I$ itself is not a maximal ideal, but I don’t have a proof of that.)
The quotient ring $A/\mathfrak{m} = K[\{t_{f,i}\}]/\mathfrak{m}$ is a field and the natural composite homomorphism $K \to A \to A/\mathfrak{m}$ of rings let us view the field $A/\mathfrak{m}$ as an extension of $K$ (ring homomorphisms out of fields are always injective).

**Theorem 2.** The field $A/\mathfrak{m}$ is an algebraic closure of $K$.

**Proof.** For any nonconstant monic $f(X) \in K[X]$ we have $f(X) \equiv \prod_{i=1}^{\delta_f}(X - t_{f,i}) \in I[X] \subseteq m[X]$, so in $(A/\mathfrak{m})[X]$ we have $f(X) = \prod(X - \bar{t}_{f,i})$, where $\bar{t}_{f,i}$ denotes $t_{f,i}$ mod $m$. Each $\bar{t}_{f,i}$ is algebraic over $K$ (being a root of $f(X)$) and $A$ is generated as a ring over $K$ by the $t_{f,i}$’s, so $A/\mathfrak{m}$ is generated as a ring over $K$ by the $\bar{t}_{f,i}$’s. Therefore $A/\mathfrak{m}$ is an algebraic extension field of $K$ in which every nonconstant monic in $K[X]$ splits completely.

We will now show $A/\mathfrak{m}$ is algebraically closed, and thus it is an algebraic closure of $K$. Set $F = A/\mathfrak{m}$. It suffices to show every monic irreducible $\pi(X)$ in $F[X]$ has a root in $F$. We have already seen that any nonconstant monic polynomial in $K[X]$ splits completely in $F[X]$, so let’s show $\pi(X)$ is a factor of some monic polynomial in $K[X]$. There is a root $\alpha$ of $\pi(X)$ in some extension of $F$. Since $\alpha$ is algebraic over $F$ and $F$ is algebraic over $K$, $\alpha$ is algebraic over $K$. That implies some monic $f(X)$ in $K[X]$ has $\alpha$ as a root. The polynomial $\pi(X)$ is the minimal polynomial of $\alpha$ in $F[X]$, so $\pi(X) | f(X)$ in $F[X]$. Since $f(X)$ splits completely in $F[X]$, $\alpha \in F$. \hfill \Box

Our construction of an algebraic closure of $K$ is done, but we want to compare it with another method to put the construction in context. The idea of building an algebraic closure of $K$ by starting with a large polynomial ring over $K$ whose variables are indexed by polynomials in $K[X]$ goes back to Emil Artin. He used a large polynomial ring (somewhat smaller than the ring $K[\{t_{f,i}\}]$ we started with above) modulo a suitable maximal ideal to obtain an algebraic extension $K_1/K$ such that every nonconstant polynomial in $K[X]$ has a root in $K_1$ (not, a priori, that they all split completely in $K_1[X]$). Then he iterated this construction with $K_1$ in place of $K$ to get a new algebraic extension $K_2/K_1$, and so on, and proved that the union $\bigcup_{n \geq 1} K_n$ (or, more rigorously, the direct limit of the $K_n$’s) contains an algebraic closure of $K$ [2, pp. 544-545]. With more work, treating separately characteristic $0$ and characteristic $p$, it can be shown [3] that Artin’s construction only needs one step: $K_1$ is an algebraic closure of $K$ (so $K_n = K_1$ for all $n$, which is not obvious in Artin’s own proof). In other words, the following is true: if $F/K$ is an algebraic extension such that every nonconstant polynomial in $K[X]$ has a root in $F$ then every nonconstant polynomial in $F[X]$ has a root in $F$, so $F$ is an algebraic closure of $K$. Theorem 2 and its proof, due to B. Conrad, modifies Artin’s construction by using a larger polynomial ring over $K$ in order to adjoin to $K$ in one step a full set of roots – not just one root – of each nonconstant monic in $K[X]$, rather than one root for each polynomial. This makes it easier to prove the constructed field is an algebraic closure of $K$. A similar construction, using a maximal ideal in a tensor product, is in [1, Prop. 4, p. A V 21].

At the end of the proof of Theorem 2, the polynomial $f(X)$ in $K[X]$ with $\alpha$ as a root can be taken to be irreducible over $K$, so we could build an algebraic closure of $K$ by defining the ideal $I$ using just the monic irreducible $f(X)$ in $K[X]$ rather than all monic $f(X)$ in $K[X]$; the proofs of Lemma 1 and Theorem 2 carry over with no essential changes other than inserting the word “irreducible” in a few places. Finally, if we restrict the $f$ in the construction of $I$ to run over the monic separable polynomials in $K[X]$, or the monic separable irreducible polynomials in $K[X]$, then the field $A/\mathfrak{m}$ turns out to be a separable closure of $K$. The proof of Lemma 1 carries over with the $f_j$ being separable (or separable irreducible), and in the proof of Theorem 2 two changes are needed: $A/\mathfrak{m}$ is a separable...
algebraic extension of $K$ since it would be generated as a ring over $K$ by roots of separable polynomials in $K[X]$, and we need transitivity of separability instead of algebraicity (if $F/K$ is separable algebraic then any root of a separable polynomial in $F[X]$ is separable over $K$).

References