A $q$-ANALOGUE OF MAHLER EXPANSIONS I

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Abstract. We examine a $q$-analogue of Mahler expansions for continuous functions in $p$-adic analysis, replacing binomial coefficient polynomials $\binom{x}{n}$ with a $q$-analogue $\binom{x}{n}_q$ for a $p$-adic variable $q$ with $|q - 1|_p < 1$. Mahler expansions are recovered at $q = 1$ and we consider the $p$-adic $q$-Gamma function $\Gamma_{p,q}$ of Koblitz relative to its $q$-Mahler expansion.

1. Introduction

Let $\mathbb{Z}_p$ be the $p$-adic integers, $\mathbb{Q}_p$ the $p$-adic rationals, and $K$ a field extension of $\mathbb{Q}_p$ which is complete with respect to a nonarchimedean absolute value $|\cdot|_p$, normalized by $|p|_p = 1/p$.

About forty years ago, Mahler introduced in [18] an expansion for continuous functions from $\mathbb{Z}_p$ to $K$ using special polynomials. Specifically, he observed that the $n$th binomial coefficient polynomial

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

sends $\mathbb{Z}_p$ to $\mathbb{Z}_p$ (it sends $\mathbb{Z}$ to $\mathbb{Z} \subset \mathbb{Z}_p$, then use continuity), so $|\binom{x}{n}_p| \leq 1$ for all $x \in \mathbb{Z}_p$. Therefore for any sequence $c_n \in K$ with $\lim_{n \to \infty} c_n = 0$, the series

$$f(x) = \sum_{n \geq 0} c_n \binom{x}{n}$$

defines a continuous function $\mathbb{Z}_p \to K$. Mahler proved every continuous function from $\mathbb{Z}_p$ to $K$ arises uniquely in this way, with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k), \quad \sup_{x \in \mathbb{Z}_p} |f(x)|_p = \max_{n \geq 0} |c_n|_p.$$ 

The $c_n$ are called the Mahler coefficients of $f$ and the series $\sum c_n \binom{x}{n}$ is called the Mahler expansion of $f$.

In this paper a $q$-analogue of the Mahler expansion is studied, where $q$ is a $p$-adic variable. To set up the framework for our ideas, first we recall the philosophy of $q$-analogues over $\mathbb{R}$ and $\mathbb{C}$. For a complex number $q$ other than 1, define the $q$-analogue of a positive integer $n$ to be

$$(n)_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$ 

As $q \to 1$, $(n)_q \to n$, and this is the hallmark of a $q$-analogue: the limit as $q \to 1$ recovers the classical object. There are $q$-analogues of most functions in classical analysis [9]. For

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example, the geometric series
\[
(1 - z)^{-a} = \sum_{n \geq 0} \frac{a(a + 1) \cdots (a + n - 1)}{n!} z^n
\]
for $|z| < 1$ and $a \in \mathbb{C}$ has the $q$-anologue
\[
1 + \frac{q^a - 1}{q - 1} z + \frac{(q^a - 1)(q^{a+1} - 1)}{(q - 1)(q^2 - 1)} z^2 + \cdots = \prod_{n \geq 0} \frac{1 - q^{a+n} z}{1 - q^n z},
\]
where the infinite product converges for $|q| < 1$. The analytic treatment of $q$-series in \( \mathbb{C} \) usually assumes $|q| < 1$ or $0 < q < 1$. However, many results make sense in a formal way, allowing $q$ to be viewed as an indeterminate. The study of $q$-analogues has connections with a number of areas of mathematics, such as partitions, modular functions, and quantum groups.

The Mahler expansion in $p$-adic analysis uses binomial coefficient polynomials $\binom{x}{n}$, $x \in \mathbb{Z}_p$. For $q \in K$ with $|q - 1|_p < 1$ (the $p$-adic substitute for the condition $|q| < 1$ in $\mathbb{C}$), we will use $q$-analogues $\binom{x}{n}_q$. These are exponential functions of $x \in \mathbb{Z}_p$ if $q$ is not a root of unity, and are locally polynomials in $x$ if $q$ is a root of unity. In particular, $\binom{x}{1}_q = x$. The $q$-anologue of Mahler’s theorem is

**Theorem.** For a complete extension field $K/\mathbb{Q}_p$ and $q \in K$ with $|q - 1|_p < 1$, every continuous function $f : \mathbb{Z}_p \to K$ has a unique expansion
\[
f(x) = \sum_{n \geq 0} c_{n,q} \binom{x}{n}_q
\]
where $c_{n,q} \in K$ and $c_{n,q} \to 0$ as $n \to \infty$. Furthermore,
\[
c_{n,q} = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k), \quad \sup_{x \in \mathbb{Z}_p} |f(x)|_p = \max_{n \geq 0} |c_{n,q}|_p.
\]

About twenty years ago, van Hamme [23] proved the $p$-adic analogue of a result of F. H. Jackson on real $q$-series, thereby giving explicit polynomial approximations for continuous functions on certain compact-open subsets $V_q$ of $\mathbb{Z}_p$. The subset and the approximating polynomials depend on a parameter $q \in \mathbb{Z}_p^\times$ which can not be a root of unity. A. Verdoort has continued this work. The point of view of van Hamme and Verdoort is largely compatible with the one presented in Section 3 after a change of variables, although our approach, unlike theirs, permits a passage to the limit as $q \to 1$ to recover Mahler’s theorem at $q = 1$.

The structure of the paper is as follows. In Section 2 we review some properties of $q$-analogues, where $q$ will be treated mostly as an indeterminate. In Section 3 we let $q$ be a $p$-adic variable and discuss the $q$-anologue of Mahler’s theorem. Four proofs are given, having individual advantages. Because this paper may be of interest to people who work in $p$-adic analysis but not in $q$-series, and vice versa, we give extra details in Sections 2 and 3 for results that are well-known to those familiar with one of these areas but not the other.

In Section 4 we discuss properties of $q$-Mahler expansions. One aspect which is not apparent in the classical case $q = 1$ is the role of the $p$-adic logarithm in classifying differentiability in terms of $q$-Mahler expansions. In Section 5 we discuss the $q$-Mahler expansion of the $p$-adic $q$-Gamma function of Koblitz.
Here is a brief list of notation.

\( \mathbb{N} \) is the set of natural numbers \( \{0, 1, 2, \ldots \} \).

\( \mathbb{Z}_p \) is the ring of \( p \)-adic integers.

\( \mathbb{Q}_p \) is the field of \( p \)-adic numbers.

\( \zeta \) denotes a root of unity.

\( \Phi_n \) is the \( n \)th cyclotomic polynomial.

For a function \( f \) on \( \mathbb{Z}_p \), \( (E^y f)(x) = f(x+y) \) is the shift by \( y \). In particular, \( (Ef)(x) = f(x+1) \).

Let \( (K, | \cdot |) \) be a complete extension field of \( \mathbb{Q}_p \) with \( |p| = 1/p \). The continuous functions from \( \mathbb{Z}_p \) to \( K \) will be denoted \( C(\mathbb{Z}_p, K) \) and topologized by the sup-norm \( |f|_{\sup} := \sup_{x \in \mathbb{Z}_p} |f(x)| \). (We only consider \( p \)-adic absolute values, so we write \( | \cdot | \) rather than \( | \cdot |_p \).)

A function \( \mathbb{Z}_p \to K \) is called analytic if it is given by a single power series that converges on \( \mathbb{Z}_p \). It is called locally analytic if it is locally expressible by a power series around each point of \( \mathbb{Z}_p \).

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2. A Review of \( q \)-Formalism

Here we recall the features of \( q \)-analogues that are needed for our purposes, generally insofar as \( q \) can be treated as an indeterminate. Some remarks will be made about specializing \( q \), especially at roots of unity. The focus will be on properties of \( q \)-binomial coefficients and \( q \)-difference operators.

For an integer \( n \) and an indeterminate \( q \), the \( q \)-analogue of \( n \) is

\[
(n)_q := \frac{q^n - 1}{q - 1}.
\]

For example, \((0)_q = 0, (1)_q = 1, (2)_q = 1 + q, (-1)_q = -1/q \).

When \( n \geq 1 \), \((n)_q = 1 + q + \cdots + q^{n-1}\) is a polynomial in \( \mathbb{Z}[q] \).

For any integers \( m \) and \( n \),

\[(2.1) \quad (-n)_q = -\frac{1}{q^n} (n)_q, \quad (n)_{1/q} = \frac{1}{q^{n-1}} (n)_q, \quad (mn)_q = (m)_q (n)_q (q^n) .\]

Specializing \( q = 1 \), \((n)_q \) becomes \( n \).

The \( q \)-factorials are

\[
(n)_q! := \begin{cases} 
1, & n = 0; \\
(n)_q (n-1)_q \cdots (1)_q, & n \geq 1.
\end{cases}
\]

For example, \((1)_q! = 1, (2)_q! = 1 + q, (3)_q! = 1 + 2q + 2q^2 + q^3 \), and

\[(2.2) \quad (n)_{1/q}! = \frac{1}{q^{n(n-1)/2}} (n)_q! .\]
The $q$-binomial coefficient for nonnegative integers $m$ and $n$ with $m \geq n$ is
\[
\binom{m}{n}_q := \frac{(m)_q!}{(n)_q!(m-n)_q!} = \frac{(m)_q(m-1)_q \cdots (m-n+1)_q}{(n)_q!} = \frac{(q^n-1)(q^{n-1}-1) \cdots (q^{m-n+1}-1)}{(q^n-1)(q^{n-1}-1) \cdots (q-1)}.
\]
We use the second or third expression to extend the definition of $\binom{m}{n}_q$ to any integer $m$.
These functions go back to Gauss [10, p. 16], so they are also called Gaussian coefficients.

The first few $q$-binomial coefficients are
\[
\binom{m}{0}_q = 1, \quad \binom{m}{1}_q = (m)_q = \frac{q^m - 1}{q - 1}, \quad \binom{m}{2}_q = \frac{(q^m - 1)(q^{m-1} - 1)}{(q^2 - 1)(q - 1)}.
\]
For $m \geq n$, $\binom{m}{n}_q = \binom{m-n}{n}_q$, and as a rational function in $q$, $\binom{m}{n}_q = 0$ precisely when $0 \leq m < n$. The $q$-binomial coefficient may vanish in other cases numerically, e.g., $(\frac{2}{3})_q = (1 + q^2)(1 + q + q^2)$, so $\binom{2}{3}_q = 0$.

The following result is essentially due to Gauss [10, p. 17].

**Theorem 2.1.** For fixed integers $m \geq n \geq 0$, $\binom{m}{n}_q \in \mathbb{Z}[q]$ with degree $n(m-n)$.

**Proof.** The degree follows from the definition, once we know $\binom{m}{n}_q$ is a polynomial in $q$.

We give Gauss’ proof that $\binom{m}{n}_q \in \mathbb{Z}[q]$ and then an alternate proof that seems to be new.

The Pascal’s triangle recursion for binomial coefficients generalizes (for all $m$ in $\mathbb{Z}$) to
\[
\binom{m}{n}_q = \binom{m-1}{n-1}_q + q^n \binom{m-1}{n}_q = q^{m-n} \binom{m-1}{n-1}_q + \binom{m-1}{n}_q
\]
(when $m \geq n$, replace $n$ by $m - n$ to obtain either recursion from the other), and iterating the second recursion gives
\[
\binom{m+n+1}{n+1}_q = q^m \binom{m+n}{n}_q + \binom{m+n}{n+1}_q = \sum_{k=0}^{m} q^k \binom{k+n}{n}_q.
\]
So $\binom{m}{n}_q \in \mathbb{Z}[q]$ by induction on $n$ (and actually all the coefficients are nonnegative).

As an alternate proof, the irreducible factors of the rational function $\binom{m}{n}_q$ are cyclotomic polynomials. The multiplicity of the $j$th cyclotomic polynomial $\Phi_j(q)$ as a factor of $(n)_q!$ is $[n/j]$, so its multiplicity as a factor of $\binom{m}{n}_q$ is $[m/j] - [n/j] - [(m-n)/j]$, which is 0 or 1. This shows for $m \geq n$ not only that $\binom{m}{n}_q$ is a polynomial in $q$, but that its irreducible factors are all simple factors and $\Phi_j(q)$ is a factor precisely when the units’ digit of $m$ in base $j$ is less than the units’ digit of $n$ in base $j$. I thank Ira Gessel for a simplification to the original form of this alternate proof.

Further identities for all $m \in \mathbb{Z}$ (and $k \geq j \geq 0$) are
\[
\binom{m}{n}_q = \frac{(m)_q(m-1)_q \cdots (m-n+1)_q}{(n)_q(n-1)_q \cdots (n-m+1)_q}, \quad \binom{m}{n}_{1/q} = \frac{1}{q^{n(m-n)}} \binom{m}{n}_q, \quad \binom{m}{k}_{q^{j}} = \binom{m}{k}_q \binom{k}{j}_q = \binom{m}{j}_q \binom{m-j}{k-j}_q.
\]
\[ \binom{-m}{n} q = (-1)^n q^{-n(n-1)/2} \binom{m+n-1}{n} q = (-1)^n \binom{m+n-1}{n}^{1/q}. \]

For example, \( \binom{-1}{n} q = (-1)^n q^{-n(n+1)/2} \). By (2.5), for \( m > 0 \), \( \binom{m}{n} q \) is a polynomial in \( 1/q \) with degree \( n(n-1)/2 + mn \) whose coefficients are nonzero integers with sign \( (-1)^n \).

The next result is a \( q \)-analogue of the binomial theorem, the \( q \)-binomial theorem. It goes back to Cauchy [4, p. 46, Eq. 18].

**Theorem 2.2.** For \( m \geq 1 \),

\[
(1 + T)(1 + qT) \cdots (1 + q^{m-1}T) = \prod_{i=0}^{m-1} (1 + q^iT) = \sum_{k=0}^{m} \binom{m}{k} q^{k(k-1)/2} T^k.
\]

Equivalently, for commuting variables \( X \) and \( Y \),

\[
(X + Y)(X + qY) \cdots (X + q^{m-1}Y) = \prod_{i=0}^{m-1} (X + q^iY) = \sum_{k=0}^{m} \binom{m}{k} q^{k(k-1)/2} X^{m-k} Y^k.
\]

**Proof.** Following Cauchy [4, p. 51], let \( h(T) = \prod_{i=0}^{m-1} (1 + q^iT) = \sum_{k=0}^{m} a_k T^k \). Then \( (1 + T)h(qT) = h(T)(1 + q^nT) \). Equating coefficients of equal powers of \( T \),

\[
a_k = \frac{q^m - q^{k-1}}{q^1 - 1}a_{k-1} = \frac{q^{m-k+1} - 1}{q^1 - 1} q^{k-1} a_{k-1},
\]

so \( a_k = \binom{m}{k} q^{k(k-1)/2} \).

In particular,

\[
(X-1)(X-q) \cdots (X-q^{m-1}) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k q^{k(k-1)/2} X^{m-k}.
\]

Actually, the idea of replacing \( T \) by \( qT \) to express \( q \)-products as \( q \)-series goes back to Euler [7, Ch. XVI, §306, 307].

The \( q^{k(k-1)/2} \) term that arises in the \( q \)-binomial theorem can be removed from explicit appearance. Define the \( n \)th \( q \)-power of a polynomial \( f(T) \) to be \( f(0; q) = 1 \) and \( f^{(n; q)} := f(T)f(qT) \cdots f(q^{n-1}T) \) for \( n \geq 1 \). Then the \( q \)-binomial theorem becomes

\[
(1 + T)^{(m; q)} = \sum_{k=0}^{m} \binom{m}{k} T^{(k; q)}.
\]

We can consider \( q \)-deformed powers of a polynomial in several variables by singling out one variable, e.g., in two variables

\[
f(X, Y)^{(m;q)} := f(X, Y)f(X, qY) \cdots f(X, q^{n-1}Y).
\]

This will appear later in the case of \( (X \pm Y)^{(m; q)} \), whose value at \( X = x, Y = y \) will be written with abuse of notation as \( (x \pm y)^{(mq)} \). For example,

\[
(x + 0)^{(n; q)} = x^n, \quad (0 + y)^{(n; q)} = q^{n(n-1)/2} y^n, \quad \binom{m}{k} q = \frac{(q^m - 1)^{(k; q)}}{(q^k - 1)^{(k; q)}}.
\]

The \( q \)-Vandermonde formula for \( \binom{m_1 + m_2}{k}_q \) is proven as for ordinary binomial coefficients.

**Theorem 2.3.** For \( m_1, m_2 \geq 0 \), \( \binom{m_1 + m_2}{k}_q = \sum_{j=0}^{k} \binom{m_1}{j} \binom{m_2}{k-j} q^{j(m_2-(k-j))} \).
Note the asymmetric roles of $j$ and $k - j$ in the exponent of $q$ on the right side.

**Proof.** Compare the coefficient of $T^k$ on both sides of

$$
\prod_{i=0}^{m_1+m_2-1} (1 + q^j T) = \prod_{i=0}^{m_2-1} (1 + q^j T) \prod_{i=0}^{m_1-1} (1 + q^j q^{m_2} T).
$$


By a specialization argument, Theorem 2.3 is true for all integers $m_1, m_2$, possibly negative.

The following simple fact will be used when we let $q$ vary $p$-adically.

**Theorem 2.4.** For $m, n \geq 0$, \( \binom{m}{n} q_1 \) \( \binom{m}{n} q_2 \) \( (q_1 - q_2) \) \( \mathbb{Z}[q_1, q_2] \).

**Proof.** For all \( i \geq 0 \), \( q_1^i - q_2^i \) \( (q_1 - q_2) \) \( \mathbb{Z}[q_1, q_2] \).

We now discuss the value of \( \binom{m}{n} q \) for \( m \geq n \) when \( q \) is specialized to various numbers.

When \( q = 1 \), \( \binom{m}{n} \) \( \binom{m}{n} \) counts the number of \( n \) element subsets of an \( m \) element set. When \( q \) is a prime power, \( \binom{m}{n} q \) counts the number of \( n \)-dimensional subspaces of an \( m \)-dimensional vector space over the field of size \( q \). This suggests the possibility of proving identities for \( q \)-binomial coefficients by letting \( q \) run through (infinitely many) prime powers and interpreting the identity as a combinatorial statement in linear algebra over finite fields. See [11] for this approach.

We now consider the case when \( q \) is specialized to a root of unity. For \( \zeta \) a root of unity of order \( b \) and \( n < b \), the value of \( \binom{m}{n} \) \( \zeta \) can be computed directly from the definition, since \( (n) \mathbb{Z} \) \( \neq 0 \). The next theorem reduces the evaluation of all \( \binom{m}{n} \) \( \zeta \) to the case when \( n < b \).

**Theorem 2.5.** Let \( \zeta \) be a root of unity of order \( b \).

i) For integers \( k \) and \( l \), with \( l \geq 0 \), \( \binom{bk}{bl} \) \( \zeta \) \( \binom{k}{l} \).

ii) For integers \( k \) and \( l \) with \( l \geq 0 \) and \( 0 \leq r, s < b \), \( \binom{bk+r}{bl+s} \) \( \zeta \) \( \binom{bk}{bl} \) \( \zeta \) \( \binom{k}{l} \) \( \zeta \) \( \binom{k}{l} \) \( \zeta \).

In particular, if \( n < b \) and \( m_1 \equiv m_2 \mod b \), then \( \binom{m_1}{n} \) \( \zeta \) \( \binom{m_2}{n} \) \( \zeta \).

**Proof.** i) \( \binom{bk}{bl} q = \prod_{j=0}^{b-1} \frac{q^{bk-j} - 1}{q^{b-l} - 1} = \prod_{j \neq 0 \mod b} \frac{q^{bk-j} - 1}{q^{b-l} - 1} \cdot \prod_{i=0}^{l-1} \frac{q^{b(k-i)} - 1}{q^{b(l-i)} - 1} \).

At \( q = \zeta \), the right side becomes \( \prod_{j=0}^{b-1} (k - i)/(l - i) = \binom{k}{l} \).

ii) First we show \( \binom{bk+a}{bl} \) \( \zeta \) \( \binom{bk+a}{bl} \) \( \zeta \) when \( a \) is not divisible by \( b \). Setting \( m = bk + a \), \( n = bl \), and \( q = \zeta \) in the equation \( \binom{m}{n} q = \binom{m}{n} q \binom{m-1}{n} q \), we get what we want. So the theorem is true for \( s = 0 \). For \( s > 1 \),

\[
\binom{bk+r}{bl+s} q = \frac{(bk+r)q(bk+r-1)q \cdots (bk+r-s+1)q}{(bl+s)(bl+s-1)q \cdots (bl+1)q} \binom{bk+r-s}{bl} q.
\]

None of the terms \( (bl+j)q \) appearing in the denominator vanishes at \( q = \zeta \), so we can evaluate and find

\[
\binom{bk+r}{bl+s} \zeta = \frac{(r)_{\zeta}(r-1)_{\zeta} \cdots (r-s+1)_{\zeta}}{(s)_{\zeta}(s-1)_{\zeta} \cdots (1)_{\zeta}} \binom{bk+r-s}{bl} q = \binom{r}{s} \binom{bk+r-s}{bl} q.
\]
Corollary 2.1. Let $\zeta$ be a root of unity of order $b$ and $n \in \mathbb{N}$. For $m$ running through a fixed residue class mod $b$, $\binom{m}{n}_\zeta$ is a polynomial in $m$.

Proof. By Theorem 2.5 (ii), $\binom{m}{n}_\zeta$ is a polynomial in $[m/b] = (m - r)/b$ and $r$ is fixed. □

Examples.

$$\binom{19}{5}_{-1} = \binom{18 + 1}{4 + 1}_{-1} = \binom{9}{2}_{1}_{-1} = 36,$$

$$\binom{17}{10}_i = \binom{16 + 1}{4 + 2}_i = \binom{4}{2}_i = 0, \quad \binom{-5}{6}_i = \binom{-8 + 3}{4 + 2}_i = \binom{-2}{1}_i = -2i.$$

The periodicity of $\binom{m}{n}_\zeta$ in $m$ mod $b$, stated at the end of Theorem 2.5, can also be verified by computing $\binom{m+b}{n}_\zeta - \binom{m}{n}_\zeta$ with the $q$-Vandermonde formula.

Theorem 2.5 (and an extension to $q$-multinomial coefficients) can be proven by group actions [21].

For a root of unity $\zeta$ of order $b$, that $\binom{b}{n}_\zeta = 0$ for $1 \leq n \leq b - 1$ can be seen without Theorem 2.5, since the numerator of $\binom{b}{q}_\zeta$ vanishes at $q = \zeta$ while the denominator does not, or (using Theorem 2.2) since $\prod_{j=0}^{b-1}(1 + \zeta^jT) = 1 - (-T)^b$. Stated in terms of the $b$th cyclotomic polynomial $\Phi_b(q)$, this vanishing becomes

$$\binom{b}{n}_q \equiv 0 \mod \Phi_b(q)$$

when $1 \leq n \leq b - 1$, which is also clear from the second proof of Theorem 2.1. Specializing (2.7) at $q = 1$, we recover the familiar integer congruence $\binom{p^N}{n} \equiv 0 \mod p$ when $b = p^N$ is a power of a prime $p$. Since

$$\Phi_{p^N}(q) = \frac{q^{p^N} - 1}{q^{p^N-1} - 1} = (p)_{q^{p^N-1}},$$

when $b = p^N$, (2.7) can be written as $\binom{p^N}{n}_q \equiv 0 \mod (p)_{q^{p^N-1}}$.

The $q$-analogue of the exponential series was introduced by Jackson [13]:

$$E_q(X) := \sum_{n \geq 0} \frac{X^n}{(n)_q!}.$$

(In the literature, the notation $E_q(X)$ may denote a slightly different series.) Jackson’s $q$-version of $e^{x+y} = e^x e^y$ comes from (2.2) and the $q$-binomial theorem:

$$E_q(X)E_{1/q}(Y) = \sum_{n \geq 0} \frac{(X + Y)(X + qY) \cdots (X + q^{n-1}Y)}{(n)_q!} = \sum_{n \geq 0} \frac{(X + Y)^{(n)_q}}{(n)_q!}.$$

In particular,

$$E_q(X)^{-1} = E_{1/q}(-X) = \sum_{n \geq 0} (-1)^n q^{n(n-1)/2} \frac{X^n}{(n)_q!}.$$

We now discuss $q$-difference operators. Powers $\Delta^n$ of the difference operator $\Delta$, where $(\Delta h)(x) = h(x + 1) - h(x)$ (here and in the rest of this section, $x$ is an integer variable), play a role in Mahler expansions which will taken over in the $q$-analogue by a sequence of operators $\Delta_q^n$ first introduced by Jackson [14, p. 256], [15, p. 145].
The powers of $\Delta$ behave nicely on binomial coefficients, namely
\[
\Delta^m \binom{x}{n} = \begin{cases} 
\binom{x}{n-m}, & \text{if } m \leq n; \\
0, & \text{if } m > n.
\end{cases}
\]
The $q$-analogue of powers of $\Delta$ arise naturally by considering differences of $q$-binomial coefficients.

First, note that in analogy with $\Delta \binom{x}{n} = \binom{x}{n-1}$,
\[
\Delta \binom{x}{n} = q^{x+1-n} \binom{x}{n-1}_q.
\]
Then, guided by the equation $\Delta^2 \binom{x}{n} = \Delta \binom{x+1}{n} - \Delta \binom{x}{n} = \binom{x}{n-2}$, we compute
\[
\Delta \binom{x+1}{n} = q^{x+2-n} \binom{x+1}{n-1}_q,
\]
so we’re naturally led to calculate not $\Delta \binom{x+1}{n}_q - \Delta \binom{x}{n}_q$ but
\[
\Delta \binom{x+1}{n} - q\Delta \binom{x}{n} = q^{x+2-n} \left( \binom{x+1}{n-1}_q - \binom{x}{n-1}_q \right) = q^{2(x+2-n)} \binom{x}{n-2}_q.
\]
Let $(Eh)(x) = h(x + 1)$ be the shift operator, so we’ve computed
\[
(E - I) \binom{x}{n} = q^{x+1-n} \binom{x}{n-1}_q, \quad (E - I)(E - q) \binom{x}{n} = q^{2(x+2-n)} \binom{x}{n-2}_q.
\]
Of course $n \geq 1$ and $n \geq 2$ for these respective equations.

Experience with $q$-deformed products as in the $q$-binomial theorem now makes the following definition natural: $\Delta_q^n := (E - I)^{(n;q)} = \Delta^{(n;q)}$. In full, this says
\[
\Delta_q^n := \begin{cases} 
I, & n = 0; \\
(E - I)(E - q) \cdots (E - q^{n-1}), & n \geq 1,
\end{cases}
\]
so
\[
(2.10) \quad \Delta_q^m \binom{x}{n}_q = \begin{cases} 
q^{m(x+m-n)} \binom{x}{n-m}_q, & \text{if } m \leq n; \\
0, & \text{if } m > n.
\end{cases}
\]
In particular, $\Delta_q^m \binom{x}{n}_q |_{x=0} = \delta_{mn}$. The appearance of a function of $x$ on the right side of (2.10), outside the $q$-binomial coefficient, can be removed by using an alternate $q$-difference operator:
\[
(\mathcal{D}_q^m f)(x) := q^{-mx} (\Delta_q^m f)(x).
\]
Then
\[
\mathcal{D}_q^m \binom{x}{n}_q = \begin{cases} 
q^{-m(n-m)} \binom{x}{n-m}_q, & \text{if } m \leq n; \\
0, & \text{if } m > n.
\end{cases}
\]
By (2.6),
\[
(2.11) \quad (\Delta_q^n f)(x) = \sum_{k=0}^n \binom{n}{k}_q (-1)^k q^{k(k-1)/2} f(x + n - k).
\]
The shift \( E \) commutes with multiplication by \( q \), so \( \Delta_q^n \) and \( \Delta_q^{n'} \) commute, but \( \Delta_{q^n} \Delta_{q^{n'}} \neq \Delta_{q^{n+n'}} \). To give a formula for \( \Delta_{q^{n+n'}} \) in terms of \( \Delta_{q^n} \) and \( \Delta_{q^{n'}} \),

\[
\Delta_{q^{n+n'}} = (E - q^{n+n'-1}) \cdots (E - q^{n'}) \Delta_{q^{n'}} = \sum_{k=0}^{n} \binom{n}{k}_q q^{(k(1))/2}(-q^{n'})^k E^{n-k} \Delta_{q^{n'}}
\]

by the \( q \)-binomial theorem, so

\[
(\Delta_{q^{n+n'}} f)(x) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{(k(1))/2} q^{n'k} (\Delta_{q^{n'}} f)(x + n - k) = q^{n'x} (\Delta_{q^n} g)(x),
\]

where \( g(x) = q^{-n'x} (\Delta_{q^{n'}} f)(x) \). This can be written more conveniently in terms of the \( \mathcal{D}_q^m \):

(2.12) \[ \mathcal{D}_q^{n+n'} = q^{n'n'} \mathcal{D}_q^n \mathcal{D}_q^{n'} . \]

For \( n \in \mathbb{Z} \), let \( \mathcal{U}_n(x) = q^{nx} \) (this depends on \( q \) ), so \( \Delta_q^n = \mathcal{U}_n \mathcal{D}_q^n \) and \( E^k \mathcal{U}_n = q^{kn} \mathcal{U}_n E^k \). When \( q = 1 \) the need for \( \mathcal{U}_n \) is not apparent. The notation \( \mathcal{U}_n \) comes from a similar function \( U_n \) used by Verdoort [25]. Her paper will be discussed in Section 4.

The effort to directly relate \( \Delta_q^n \Delta_q^{n'} \) with \( \Delta_{q^{n+n'}} \) led to a concise multiplicative relation (2.12) among the \( \mathcal{D}_q \)'s rather than among the \( \Delta_q \)'s. We now use (2.12) to give a formula for \( \Delta_q^n \Delta_q^{n'} \) as a linear combination of various \( \Delta_q \)'s, so the \( q \)-difference operators are a basis of the algebra they generate (they have no linear relations by (2.10)).

**Theorem 2.6.** For \( m, n \geq 0 \),

\[
\Delta_q^m \Delta_q^n = \sum_{j=0}^{m} \binom{m}{j}_q (q^n - 1)(q^n - q) \cdots (q^n - q^{m-j-1}) \Delta_q^{n+j}
\]

\[
= \sum_{j=0}^{m} \binom{m}{j}_q (q^n - 1)^{(m-j)} \Delta_q^{n+j}
\]

\[
= \sum_{i+j=m+n} \binom{m}{i}_q \binom{n}{i}_q (q^i - 1)^{(i)} \Delta_q^i.
\]

**Proof.** By the \( q \)-binomial theorem,

\[
\Delta_q^m \Delta_q^n = \sum_{k=0}^{m} \binom{m}{k}_q (-1)^{m-k} q^{(m-k)(m-k-1)/2} E^k \Delta_q^n.
\]

To get a formula for \( E^k \Delta_q^n \), we use the following identity: for all \( k \geq 0 \),

\[
a^k = \sum_{i=0}^{k} \binom{k}{i}_q (a - 1)(a - q) \cdots (a - q^{i-1}) = \sum_{i=0}^{k} \binom{k}{i}_q (a - 1)^{(i)}.
\]

This is dual to (2.6), or arises naturally from consideration of \( q \)-Mahler expansions in Section 3 (i.e., from the \( q \)-difference calculus), so we won’t stop to motivate it here. Setting \( a = E \),

(2.13) \[ E^k = \sum_{i=0}^{k} \binom{k}{i}_q \Delta_q^i . \]
Thus

\[ E^k \Delta_q^n = E^k U_n \Delta_q^n = q^{kn} U_n E^k \Delta_q^n = q^{kn} U_n \sum_{i=0}^{k} \binom{k}{i} U_i \Delta_q^i \Delta_q^n \text{ by (2.13)} \]

\[ = \sum_{i=0}^{k} \binom{k}{i} q^{n(k-i)} U_{n+i} \Delta_q^{n+i} \text{ by (2.12)} \]

\[ = \sum_{i=0}^{k} \binom{k}{i} q^{n(k-i)} \Delta_q^{n+i}, \]

so

\[ \Delta_q^m \Delta_q^n = \sum_{k=0}^{m} \sum_{i=0}^{k} (-1)^{m-k} q^{(m-k)(m-k-1)/2} q^{n(k-i)} \binom{m}{k} \binom{k}{i} \Delta_q^{n+i} \]

\[ = \sum_{i=0}^{m} \sum_{k=0}^{m-i} (-1)^{m-i-k} q^{(m-i-k)(m-i-k-1)/2} q^{nk} \binom{m-i}{k} \binom{m}{i} \Delta_q^{n+i} \]

\[ = \sum_{i=0}^{m} \binom{m}{i} q^{n-1} q^{(m-i)q} \Delta_q^{n+i} \text{ by (2.6)} \]

\[ = \sum_{i=0}^{m} \binom{m}{i} q^{n-1} q^{i} \Delta_q^{m+n-i}. \]

\[ \square \]

**Example.** \( \Delta_q^2 \Delta_q^n = (q^n - 1)(q^n - q) \Delta_q^n + (q^n - 1)(q + 1) \Delta_q^{n+1} + \Delta_q^{n+2}. \)

The case \( m = 1 \) of Theorem 2.6 is essentially the recursive definition \( \Delta_q^{n+1} = (E - q^n) \Delta_q^n. \)

Once the formula in Theorem 2.6 is found, it can also be proven by induction on \( m, \) without using noncommuting operators \( E^k \) and \( U_n, \) as the polynomial identity

\[ (X - 1)^{(m,q)}(X - 1)^{(n,q)} = \sum_{i=0}^{m} \binom{m}{i} q^{n-1} q^{(m-i)q}(X - 1)^{(n+i,q)} \]

\[ = \sum_{i=0}^{m} \binom{m}{i} q^{n-1} q^{(m-i)q}(X - 1)^{(m,q)}(X - q^n)^{(i,q)}. \]

Dividing by \( (X - 1)^{(m,q)} \), we get an identity which is a special case of the generalized \( q \)-binomial theorem [11, p. 252].

The \( q \)-analogue of the formula

\[ \Delta_q^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (\Delta_q^k f)(\Delta_q^{n-k} E^k g) \]
\[(2.14)\]
\[
\Delta^n_q(fg) = \sum_{k=0}^{n} \binom{n}{k} (\Delta^k_q f)(\Delta^{n-k}_q E^k g).
\]

In the inductive verification of this, use (for \(r \leq n\))
\[
(E - q^n)(FG) = (E - q^r)F \cdot EG + q^r F \cdot (E - q^{n-r})G
\]
with \(F = \Delta^k_q f, G = \Delta^{n-k}_q E^k g\), and \(r = k\).

3. \(p\)-adic Features of \(q\)-Formalism

In Section 2, the emphasis was on \(q\) as an indeterminate. Here it will be on \(q\) as a \(p\)-adic variable, i.e., as an element of a complete valued field \(K\) containing \(\mathbb{Q}_p\). (We do not assume \(q \in \mathbb{Q}_p\).) As we will have no use for the archimedean absolute value function, the absolute value on \(K\) will be denoted simply as \(| \cdot |\), and \(\text{ord}\) is the corresponding additive valuation: \(|z| = (1/p)^{\text{ord}(z)}\). The valuation ring \(\{z \in K : |z| \leq 1\}\) will be denoted \(\mathcal{O}_K\), with maximal ideal \(m_K\). We normalize the absolute value so \(|p| = 1/p\).

For the benefit of readers outside of number theory, we recall some facts about power functions and roots of unity in \(p\)-adic fields.

**Lemma 3.1.** (i) The roots of unity in \(K\) which reduce to 1 in the residue field \(\mathcal{O}_K/m_K\) are exactly the \(p\)th power roots of unity in \(K\).

(ii) If \(\zeta\) is a root of unity of order \(p^N \geq 1\), then
\[
|\zeta - 1| = (1/p)^{1/p^{N-1}(p-1)} \geq (1/p)^{1/(p-1)}.
\]

The roots of unity in \(K\) are a discrete set.

(iii) For \(q \in K\), the sequence \(\{1, q, q^2, q^3, \ldots\}\) can be extended to a continuous function \(q^x\) for \(x \in \mathbb{Z}_p\) if and only if \(|q - 1| < 1\), in which case
\[
q^x = \sum_{n \geq 0} (q - 1)^n \binom{x}{n}, \quad |q^x - 1| \leq |q - 1| < 1.
\]

(iv) If \(|q - 1| < 1\), then \(q^x = 1\) for \(x \neq 0\) if and only if \(q\) is a root of unity of order \(p^N\) and \(x \in p^N \mathbb{Z}_p\).

**Proof.** (i) The residue field \(\mathcal{O}_K/m_K\) has characteristic \(p\). Since \(X^a - 1\) has distinct roots in characteristic \(p\) when \(a\) is prime to \(p\), a root of unity \(\zeta\) in \(K\) of order \(ap^b\) with \(a > 1\) and \((a, p) = 1\) has \(\zeta^{p^b} \not\equiv 1 \mod m_K\), so \(\zeta \not\equiv 1 \mod m_K\). Since the only \(p\)th power root of unity in characteristic \(p\) is 1, if \(\zeta^{p^N} = 1\) in \(K\), then in the residue field of \(K\) we have \(\zeta^{p^N} \equiv 1 \mod m_K\), so \(\zeta \equiv 1 \mod m_K\).

(ii) We have
\[
\prod_{i=1 \atop (p,i)=1}^{p^N} (1 - \zeta^i) = \Phi_{p^N}(1) = p,
\]
so
\[
p = (1 - \zeta)^{p^{N-1}(p-1)} \prod_{i=1 \atop (p,i)=1}^{p^N} \frac{1 - \zeta^i}{1 - \zeta}.
\]
and for \(i\) prime to \(p\), the ratio \((1 - \zeta^i)/(1 - \zeta) = 1 + \zeta + \cdots + \zeta^{i-1}\) is congruent in the residue field of \(K\) to \(i \neq 0 \mod m_K\), so this ratio has absolute value 1, hence \(1 - \zeta\) has the indicated size.

For two distinct roots of unity \(\zeta\) and \(\zeta'\) in \(K\), either \(\zeta \not\equiv \zeta' \mod m_K\), so \(|\zeta - \zeta'| = 1\), or \(\zeta/\zeta' \equiv 1 \mod m_K\), and then \(|\zeta - \zeta'| = |\zeta/\zeta' - 1| \geq (1/p)^{1/(p-1)}\), so the roots of unity in \(K\) are a (bounded) discrete set.

(iii) For “if”, we have for any \(m \in \mathbb{N}\) that

\[
q^m = (1 + q - 1)^m = \sum_{n=0}^{m} (q - 1)^n \binom{m}{n}.
\]

Since \((q - 1)^n \to 0\), the continuous function

\[
q^x = \sum_{n \geq 0} (q - 1)^n \binom{x}{n}
\]
on \(\mathbb{Z}_p\) is the \(p\)-adic interpolation of \(\{q^m\}_{m \geq 0}\). For “only if”, \(q^p \to q^0 = 1\) as \(N \to \infty\), so \(|q| = 1\) and as in (i) we conclude \(|q - 1| < 1\).

(iv) Let \(x = p^n u\) with \(u\) a unit in \(\mathbb{Z}_p\). Then \(q^{p^n u} = 1\) if and only if \(q^{p^n} = 1\), by taking the \((1/u)\)th power.

Applying (iii) to \(q\)-analogues, \((m)_q = (q^m - 1)/(q - 1)\) for \(m \in \mathbb{Z}\) extends to a continuous function \((x)_q\) for \(x \in \mathbb{Z}_p\) if and only if \(|q - 1| < 1\), in which case the extension to \(\mathbb{Z}_p\) is

\[
(x)_q = \begin{cases} 
\frac{q^{x-1}-1}{q-1}, & \text{if } q \neq 1; \\
x, & \text{if } q = 1,
\end{cases}
\]

and by (iii), \((x)_q \equiv x \mod m_K\). In particular, if \(x \in \mathbb{Z}_p^\times\), then \((x)_q \in O_K^\times\).

For \(q \neq 1\), \((x)_q\) is a nonvanishing function unless, by (iv), \(q\) is a nontrivial root of unity of order \(p^N\), in which case \((x)_q = (j)_q\) where \(x \equiv j \mod p^N\) and \(0 \leq j \leq p^N - 1\).

We now define the \(q\)-analogue of binomial coefficient functions.

For \(|q - 1| < 1\), \((-m)_q\) has a continuous extension from \(m \in \mathbb{Z}\) to \(x \in \mathbb{Z}_p\), given by

\[
\binom{x}{n}_q = \frac{(x)_q(x-1)_q \cdots (x-n+1)_q}{(n)_q!}
= \frac{(q^x-1)(q^{x-1}-1) \cdots (q^{x-m+1}-1)}{(q^n-1)(q^{n-1}-1) \cdots (q-1)},
\]

provided \((n)_q! \neq 0\), i.e., \(q\) is not a nontrivial \(p\)th power root of unity of order \(\leq n\).

If \(|q-1| < 1\) and \(q\) is a root of unity of order \(p^N\), Corollary 2.1 implies \((-m)_q\) is a polynomial function of \(x\) on cosets of \(p^N\mathbb{Z}_p\). For \(x = p^N y + r\) and \(n = p^N l + s\) where \(0 \leq r, s < p^N\), Theorem 2.5(ii) extends by continuity to

\[
\binom{x}{n}_q = \binom{y}{l}_q \binom{r}{s}_q
\]

(3.1)

For example, if \(p = 2\), then

\[
\binom{x}{2l}_1 = \begin{cases} 
(x/2)_l, & \text{if } x \equiv 0 \mod 2; \\
(x-1)_l/2, & \text{if } x \equiv 1 \mod 2,
\end{cases}
\]

\[
\binom{x}{2l+1}_1 = \begin{cases} 
0, & \text{if } x \equiv 0 \mod 2; \\
((x-1)/2)_l, & \text{if } x \equiv 1 \mod 2.
\end{cases}
\]
So \( \binom{\cdot}{n}_q \) is an exponential function of \( x \) (a polynomial in \( q^x \)) if \( q \) is not a root of unity and is locally a polynomial in \( x \) if \( q \) is a root of unity.

By Theorem 2.1, \( |\binom{\cdot}{n}_q| \leq 1 \) for all \( x \in \mathbb{Z}_p \), with equality if \( x = n \).

The difference operators \( \Delta^n_q \) and \( \Omega^n_q \) make sense on functions of a p-adic integer variable \( x \), and equations (2.10) and (2.11) remain true when \( x \) is any \( p \)-adic integer.

By continuity, Theorems 2.3 and 2.4 become

**Theorem 3.1.** If \( |q - 1| < 1 \), then for all \( x, y \in \mathbb{Z}_p \), \( \binom{x+y}{k}_q = \sum_{j=0}^{k} \binom{x}{j}_q \binom{y}{k-j}_q q^{j(y-(k-j))} \).

**Theorem 3.2.** If \( x \in \mathbb{Z}_p \) and \( |q_1 - 1| < 1, |q_2 - 1| < 1 \), then \( |\binom{\cdot}{n}_q| - \binom{\cdot}{n}_{q_1} |q_2| \leq |q_1 - q_2| \).

So \( \binom{\cdot}{n}_q = \lim_{q \to q} \binom{\cdot}{n}_{q'} \). In particular, formulas involving \( q \)-binomial coefficients when \( q \) is a root of unity can be computed first at non roots of unity and then pass to a limit.

For example, let \( 1 \leq k \leq p^r \) with \( k = p^j k' \) and \( k' \) prime to \( p \). For \( |q - 1| < 1 \) with \( q \) not a root of unity,

\[
\binom{p^r}{k}_q = \binom{p^r}{k}_q \binom{p^r - 1}{k - 1}_q = \binom{p^r}{k}_q \frac{1}{(k')_{q^{\omega(q)}}} \binom{p^r - 1}{k - 1}_q.
\]

In \( \mathcal{O}_K/m_K \), \( \binom{p^r - 1}{k - 1}_q \equiv \binom{p^r - 1}{k - 1}_q \equiv (-1)^{k-1} \) and \( (k')_{q^{\omega(q)}} \equiv k' \neq 0 \), so

\[
(3.2) \quad \binom{p^r}{k}_q = |\binom{p^r}{k}_q|.
\]

By continuity in \( q \), (3.2) is also true when \( q \) is a root of unity. Alternatively, (3.1) could be used instead for a direct calculation when \( q \) is not a root of unity.

We now discuss the \( q \)-analogue of Mahler expansions.

**Theorem 3.3 (\( q \)-Mahler Theorem).** For \( q \in K \) with \( |q - 1| < 1 \), every continuous function \( f: \mathbb{Z}_p \to K \) has a unique representation in the form

\[
f(x) = \sum_{n \geq 0} c_{n,q} \binom{x}{n}_q
\]

where \( c_{n,q} \in K \) and \( \lim_{n \to \infty} c_{n,q} = 0 \). A formula for \( c_{n,q} \) is

\[
c_{n,q} = (\Delta^n_q f)(0) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{k} q^{k(k-1)/2} f(n - k) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k).
\]

We will give four proofs of Theorem 3.3 below.

In Theorem 3.3, we call \( c_{n,q} \) the \( n \)th \( q \)-Mahler coefficient of \( f \) and \( \sum c_{n,q} \binom{x}{n}_q \) the \( q \)-Mahler expansion of \( f \). The terms “Mahler coefficient” and “Mahler expansion” will refer to the case \( q = 1 \). The formula for \( c_{n,q} \) in Theorem 3.3 will be called the \( q \)-Mahler Inversion Formula.

The formula for \( c_{n,q} \) follows from computing \( (\Delta^n_q f)(0) \) using (2.10). Replacing \( f \) by \( (E^y f)(x) = f(x + y) \), we have \( \lim_{n \to \infty} (\Delta^n_q f)(y) = 0 \) for all \( y \in \mathbb{Z}_p \). Like the case \( q = 1 \), this limit turns out to be uniform in \( y \), and in fact there is some uniformity in \( q \) as well (which
is not apparent by looking only at the case \( q = 1 \). Such uniformities will arise from two of the proofs of Theorem 3.3.

**Example.** For \(|a - 1| < 1\) and \(|q - 1| < 1\),

\[
(3.3) \quad a^x = \sum_{n \geq 0} (a - 1)(a - q) \cdots (a - q^{n-1}) \binom{x}{n}_q = \sum_{n \geq 0} (a - 1)^{(n,q)} \binom{x}{n}_q.
\]

**Example.** Using the \( q \)-binomial theorem, the sequence \((1 + t)^{(m,q)}\) extends continuously from \( m \in \mathbb{N} \) to \( x \in \mathbb{Z}_p \) if and only if \(|t| < 1\), when

\[
(1 + t)^{(x,q)} = \sum_{n \geq 0} q^{n(n-1)/2} \binom{x}{n}_q.
\]

This could also be proven in a style similar to that of Lemma 3.1 (iii).

For any \( x, y \in \mathbb{Z}_p \), \((1 + t)^{(x+y,q)} = (1 + t)^{(x,q)}(1 + q^x t)^{(y,q)}\). Setting \( y = -x \) yields

\[
(1 + t)^{(r,q)} = (1 + q^r t)^{(-r,q)}.
\]

For example, computing \((1 + q^m t)^{(-m,q)}\) in two ways for \( m \geq 1 \), we have

\[
\frac{1}{(1 + t)(1 + qt) \cdots (1 + q^{m-1} t)} = \sum_{n \geq 0} q^{n(n-1)/2} (q^n t)^n \binom{-m}{n}_q = \sum_{n \geq 0} \binom{m + n - 1}{n}_q (-t)^n,
\]

which is due to Cauchy [4, Eq. 19, p. 46] as an identity over the complex numbers.

**Warning.** For \(|a - 1| < 1\), writing \( a = 1 + t \), it seems reasonable to define \( a^{(x,q)} = (1 + t)^{(x,q)} \) in the sense of the above example. However, although \(|a^{(m,q)} - 1| < 1\) and \((1 + T)^{(mn,q)} = ((1 + T)^{(m,q)})^{(n,q^m)}\) (which implies (2.1) by looking at the coefficient of \( T \)), it is false that \( a^{(mn,q)} = (a^{(m,q)})^{(n,q^m)}\), even when \( m = n = 2 \). A correct way to state the \( q \)-version of \((1 + T)^{mn} = ((1 + T)^{m})^{n}\) so that it is valid to specialize the variable is

\[
(1 + T)^{(mn,q)} = (1 + T)^{(n,q^m)}(1 + qT)^{(n,q^m)} \cdots (1 + q^{m-1}T)^{(n,q^m)}.
\]

Our first proof of Theorem 3.3 will deduce the result from the known case \( q = 1 \). Recall that a countable set of vectors \( \{e_n\}_{n \geq 0} \) in a \( K \)-Banach space \((V, \| \cdot \|)\) (we assume the norm on \( V \) is nonarchimedean: \(|v + w| \leq \max(|v|, |w|)|\)) is called an orthonormal basis if every \( v \in V \) has a unique representation in the form \( v = \sum c_n e_n \) where \( c_n \to 0 \) and \( \|v\| = \max |c_n| \). Mahler’s theorem says the functions \((e_n^*)\) are an orthonormal basis of \( C(\mathbb{Z}_p, K)\), topologized by the sup-norm.

The following standard lemma shows that a small perturbation of an orthonormal basis is still an orthonormal basis. The ideas in the proof are taken from [3, Prop. 2 §1.1.4, Prop. 4 §2.7.2].

**Lemma 3.2.** Let \( K \) be a complete nonarchimedean nontrivially valued field and \( V \) be a \( K \)-Banach space with an orthonormal basis \( \{e_n\}_{n \geq 0} \). If \( e'_n \in V \) with \( \sup_{n \geq 0} \|e_n - e'_n\| < 1 \), then \( \{e'_n\} \) is an orthonormal basis of \( V \).

**Proof.** Step 1) \( \| \sum_{n=0}^N c_n e'_n\| = \max_{0 \leq n \leq N} |c_n| \).
Let \( \varepsilon = \sup_{n \geq 0} ||e_n - e'_n|| < 1 \). Writing

\[
\sum_{n=0}^{N} c_n e'_n = \sum_{n=0}^{N} c_n (e'_n - e_n) + \sum_{n=0}^{N} c_n e_n,
\]

the first sum has size at most \( \varepsilon \max |c_n| \).

Step 2) The \( K \)-linear span (= finite linear combinations) of the \( e'_n \) is dense in \( V \).

Let \( W \) be this span. For \( v \in V \), let \( v = \sum_{n \geq 0} c_n e_n \). Choose \( N \) so \( |c_n| \leq \varepsilon \|v\| \) for \( n \geq N + 1 \). Then

\[
v - \sum_{n=0}^{N} c_n e'_n = \sum_{n=0}^{N} c_n (e_n - e'_n) + \sum_{n=N+1}^{\infty} c_n e_n
\]

has norm \( \leq \varepsilon \|v\| \). Assume \( W \) is not dense, so there is \( v \in V \) such that \( a = \inf_{w \in W} \|v - w\| > 0 \). Since \( a/\varepsilon > a \), there is \( w \in W \) such that \( 0 < \|v - w\| < a/\varepsilon \). From above, there is \( w' \in W \) such that

\[
\|v - w - w'\| \leq \varepsilon \|v - w\| < a,
\]

a contradiction.

Step 3) \( \{e'_n\} \) is an orthonormal basis.

By Step 1, it suffices to show for each \( v \in V \) that \( v = \sum c_n e'_n \) for some sequence \( c_n \to 0 \) in \( K \).

Choose \( w_1 \in W \) such that \( \|v - w_1\| \leq 1/2 \). Choose \( w_2 \in W \) such that \( \|v - w_1 - w_2\| \leq 1/4 \).

Continuing, choose \( w_m \in W \) such that \( \|v - w_1 - \cdots - w_m\| \leq 1/2^m \).

Then \( \|w_m\| \to 0 \) and \( v = \sum w_m \).

Writing \( w_m = \sum b_{m,n} e'_n \), we have \( b_{m,n} = 0 \) for \( n \) large and \( |b_{m,n}| \leq \|w_m\| \) by Step 1. Thus

\[
v = \sum_{m} \left( \sum_{n} b_{m,n} e'_n \right) = \sum_{m} \left( \sum_{n} b_{m,n} \right) e'_n,\]

where the interchange of the double sum is justified by [12, Lemma 4.1.3].

Here is a first proof of Theorem 3.3.

**Proof.** By Mahler’s theorem, \( \{\binom{x}{n}\}_{n \geq 0} \) is an orthonormal basis of \( C(\mathbb{Z}_p, K) \). For all \( n \geq 0 \), Theorem 3.2 implies

\[
\left| \binom{x}{n}_q - \binom{x}{n} \right|_{\sup} \leq |q - 1| < 1.
\]

Therefore we are done by Lemma 3.2.

This proof of Theorem 3.3 is succinct, but depends on already having the result in the case \( q = 1 \). The same argument would deduce the result for all \( q \) with \( |q - 1| < 1 \) if we had it for any one such \( q \).

By a similar idea, since \( \{\binom{x}{m}
binom{y}{n}\}_{m,n} \) is an orthonormal basis of \( C(\mathbb{Z}_p \times \mathbb{Z}_p, K) \), topologized by the sup-norm, so is \( \{\binom{x}{m} \binom{y}{n} \}_{m,n} \) for fixed \( q_1, q_2 \in K \) with \( |q_1 - 1|, |q_2 - 1| < 1 \). There is a similar extension to \( C(\mathbb{Z}_p^r, K) \) for any \( r \geq 1 \).

Since \( \Delta_q^n f(x) = \Delta^n_q (E^x f)(0) \), by the \( q \)-Mahler theorem we have \( \lim_{n \to \infty} \Delta_q^n f(x) = 0 \) for each \( x \in \mathbb{Z}_p \). However, this limit is actually uniform in \( x \). To see this we give a second proof of the \( q \)-Mahler theorem, one which will not assume Mahler’s theorem already. It will show directly that \( \lim_{n \to \infty} \Delta_q^n f = 0 \) in \( C(\mathbb{Z}_p, K) \).
First we record a lemma. It gives some properties of the size of \((x)_q\). Extending (2.1) from \(Z\) to \(Z_p\), if \(|q - 1| < 1\) then \((xy)_q = (x)_q(y)_q^p\) for \(x, y \in Z_p\). In particular, for \(n \in \mathbb{N}\) and \(u \in Z_p^\times\),

\[
(p^n u)_q = (p^n)_q(u)_q^p.
\]

**Lemma 3.3.** Let \(|q - 1| < 1\).

(i) If \(x = p^n u\) with \(u \in Z_p^\times\), \(|(x)_q| = |(p^n)_q|\).

(ii) \(|(p^n)_q| \leq \prod_{k=0}^{n-1} \max(|q^k - 1|, 1/p) \leq \max(|q - 1|, 1/p)^n < 1\).

(iii) If \(|q - 1| < (1/p)^{1/(p-1)}\), then \(|(x)_q| = |x|\) for all \(x \in Z_p\).

**Proof.** (i) Use (3.4), recalling \((u)_q^p = u \equiv 0 \mod m_K\).

(ii) By (2.1),

\[
(p^n)_q = (p)_q(p)(p)_q^p \cdots (p)_q^p = (p)_q^q = \Phi_{p}(q) \equiv (q - 1)^{p-1} \equiv 0.
\]

(iii) By (i), we only need to show the result for \(x = p^n\). Moreover, by (3.5) and \(|q^k - 1| \leq |q - 1| < (1/p)^{1/(p-1)}\), it suffices to show the result for \(x = p\). Since

\[
(p)_q = q^p - 1 = \sum_{k=1}^{p} \binom{p}{k}(q - 1)^{k-1}
\]

and each term in the sum except the one for \(k = 1\) has size less than \(1/p\), we’re done. \(\square\)

As a consequence of (i) and (ii), we have

\[
|(x)_q - (y)_q| = |(x - y)_q| \leq \max(|q - 1|, 1/p)^{\text{ord}(x-y)},
\]

which can be rewritten as \(|q^x - q^y| \leq |q - 1| \max(|q - 1|, 1/p)^{\text{ord}(x-y)}\), in which form it appears in [22, Theorem 32.4].

From (ii), (3.2) can be weakened to

\[
\left| \binom{p^r}{k}_q \right| \leq \max(|q - 1|, 1/p)^{r-j},
\]

where we recall \(1 \leq k \leq p^r, j = \text{ord}(k)\).

We now give a second proof of Theorem 3.3. The idea is taken from the proof of Mahler’s theorem in [22, Exer. 52.E].

**Proof.** Since \(|\Delta_q^{n+1} f|_{\text{sup}} \leq |\Delta_q^n f|_{\text{sup}}\), it suffices to show \(\lim_{r \to \infty} \Delta_q^{p^r} f = 0\). We have

\[
(\Delta_q^{p^r} f)(x) = \sum_{k=0}^{p^r} \binom{p^r}{k}_q (-1)^{p^r-k} q^{(p^r-k)(p^r-k-1)/2} f(k + x)
\]

\[
= \sum_{k=0}^{p^r} \binom{p^r}{k}_q (-1)^{p^r-k} q^{(p^r-k)(p^r-k-1)/2} (f(k + x) - f(x)).
\]

The \(k = 0\) term vanishes, so by (3.6)

\[
|\Delta_q^{p^r} f|_{\text{sup}} \leq \max_{i+j=r} \max(|q - 1|, 1/p)^{i+j} \rho_j(f),
\]

where \(\rho_j(f) = \max_{i+j=r} \max(|q - 1|, 1/p)^{i+j} \rho_j(f)\).
where $\rho_j(f) = \sup_{|x-y| \leq 1/p^j} |f(x) - f(y)|$. The terms indexed by $i$ and $j$ are both uniformly bounded above, and each tends to zero for large values of the index.

Not only does this show $\lim_{n \to \infty}(\Delta^q_n f)(x) = 0$ uniformly in $x$, but also (for fixed $\delta \in (0, 1)$) uniformly in $q$ for $|q-1| \leq \delta < 1$.

For the third proof of the $q$-Mahler theorem, we extend a periodicity property of ordinary binomial coefficients to $q$-binomial coefficients: for any $N \geq 1$ and all $n < p^N$,

$$a \equiv b \mod p^N \Rightarrow \left(\frac{a}{n}\right) \equiv \left(\frac{b}{n}\right) \mod p.$$  

For $q$-binomial coefficients, the same result is true provided $N$ is taken large enough depending on $q$.

**Lemma 3.4.** Let $|q-1| < 1$. For $N$ large, depending on $q$, if $x \equiv y \mod p^N \mathbb{Z}_p$ and $n < p^N$ then

$$\left|\left(\frac{x}{n}\right)_q - \left(\frac{y}{n}\right)_q\right| \leq \frac{1}{p}.$$  

More precisely, this is true if $1/(p^{N-1}(p-1)) < \text{ord}(q-1)$.

**Proof.** By Theorem 2.5,

$$m_1 \equiv m_2 \mod p^N \Rightarrow \left(\frac{m_1}{n}\right)_q - \left(\frac{m_2}{n}\right)_q \in \Phi_{p^N}(q)\mathbb{Z}[q].$$

So by continuity,

$$\left|\left(\frac{x}{n}\right)_q - \left(\frac{y}{n}\right)_q\right| \leq \left|\Phi_{p^N}(q)\right| = \left|(p)^{q^{N-1}}\right|.$$  

For $N$ large, $|q^{p^N-1} | < (1/p)^{1/(p-1)}$, so $(p)^{q^{N-1}}$ has size $|p| = 1/p$ by Lemma 3.3(iii).

Let’s be more precise about how large $N$ has to be. For any $N$,

$$\Phi_{p^N}(q) = \prod_{\zeta \neq 1, \zeta^{p^N-1}} (q - \zeta).$$

There are $p^{N-1}(p-1)$ terms in the product. When $1/p^{N-1}(p-1) < \text{ord}(q-1)$, then $|q-1| < |\zeta - 1| = (1/p)^{1/p^{N-1}(p-1)}$ for all such $\zeta$ by Lemma 3.1(ii), so all the terms have the same size and therefore

$$\left|\Phi_{p^N}(q)\right| = \frac{1}{p}.$$  

If we work modulo $(q-1, p)$, then for $x \equiv y \mod p^N$ and $n < p^N$, $\left(\frac{x}{n}\right)_q \equiv \left(\frac{y}{n}\right)_q \equiv (\frac{y}{n})_q$, so without needing $N$ to be large, we have $\left|\left(\frac{x}{n}\right)_q - \left(\frac{y}{n}\right)_q\right| \leq \max(|q-1|, 1/p)$.

Now we give a third proof of Theorem 3.3. Like the second, it does not require prior knowledge at $q = 1$. It is based on the proof in [17, pp. 99–100].

**Proof.** Let

$$L: \{(c_n)_{n \geq 0} : c_n \in K, c_n \to 0\} \to C(\mathbb{Z}_p, K)$$
by $(c_n) \mapsto \sum_{n \geq 0} c_n \binom{x}{n}_q$. This is $K$-linear and continuous, where the domain and range are both topologized by the appropriate sup-norm. We want to show $L$ is onto. By scaling it suffices to show the restriction $L : B \to C(\mathbb{Z}_p, \mathcal{O}_K)$ is onto, where

$$B = \{(c_n) : |c_n| \leq 1, c_n \to 0\}.$$ 

By completeness of $B$ and continuity of $L$, it is enough to show that for any $f \in C(\mathbb{Z}_p, \mathcal{O}_K)$, there is some $s \in B$ such that $|f - L(s)| \leq |p|$. (Then apply the result to $g = (f - L(s))/p$ to get $s' \in B$ such that $|f - L(s + ps')| \leq |p^2|$, etc.) That is, we want to show surjectivity of the map

$$\{(c_n) : c_n \in \mathcal{O}_K/p, c_n = 0 \text{ for large } n\} \to C(\mathbb{Z}_p, \mathcal{O}_K/p)$$

given by

$$(c_n) \mapsto \sum_{n \geq 0} c_n \binom{x}{n}_q \mod p.$$ 

Note that the quotient topology on $\mathcal{O}_K/p$ is the discrete topology. Thus

$$C(\mathbb{Z}_p, \mathcal{O}_K/p) = \bigcup_{N \geq 1} \text{Maps}(\mathbb{Z}_p/p^N \mathbb{Z}_p, \mathcal{O}_K/p).$$

The union in (3.8) can be taken over just large integers. Lemma 3.4 suggests that at least for large $N$ (depending on $q$), $f \in C(\mathbb{Z}_p, \mathcal{O}_K/p)$ factors through $\mathbb{Z}_p/p^N \mathbb{Z}_p$ when its $n$th $q$-Mahler coefficient vanishes for $n \geq p^N$, thus suggesting the more precise surjectivity of

$$\{(c_n)_{n=0}^{p^N-1} : c_n \in \mathcal{O}_K/p\} \to \text{Maps}(\mathbb{Z}_p/p^N \mathbb{Z}_p, \mathcal{O}_K/p)$$

given by (3.7) with the sum over $0 \leq n \leq p^N - 1$. (Note that by Lemma 3.4, $(x)_{n} \mod p$ is well-defined on $\mathbb{Z}_p/p^N \mathbb{Z}_p$ for $N$ large and $n < p^N$.) The surjectivity (even bijectivity) of (3.9) follows from the argument that $q$-Mahler coefficients are unique.

We could have worked in $\mathcal{O}_K/(q-1, p)$ and not needed to use only large $N$ at the end of the proof.

Here’s a fourth proof of Theorem 3.3, which like the second will yield some uniformity statements in $q$.

**Proof.** Define the numbers $c_n = c_{n, q}$ as in the statement of Theorem 3.3, so

$$f(m) = \sum_{n \geq 0} c_n \binom{m}{n}_q$$

for all nonnegative integers $m$. We thus only need to show that $|c_n| \to 0$. To do this we adapt Bojanic’s argument in [2].

Bojanic’s proof uses two different formulas for $(\Delta^nf)(m)$. First,

$$(\Delta^nf)(m) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k + m).$$

Writing $(\Delta^nf)(m) = (\Delta^nE^m f)(0)$, we also have

$$E^m = (I + \Delta)^m = \sum_{j=0}^{m} \binom{m}{j} \Delta^j \Rightarrow (\Delta^nf)(m) = \sum_{j=0}^{m} \binom{m}{j} (\Delta^{n+j}f)(0).$$
For the $q$-analogue of these, (2.11) gives
\[
(\Delta_q^n f)(m) = \sum_{k=0}^{m} \binom{n}{k} (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k + m),
\]
while (2.13) gives
\[
E^m = \sum_{j=0}^{m} \binom{m}{j} \Delta_q^j \Rightarrow (\Delta_q^n f)(m) = (\Delta_q^n E^m f)(0) = \sum_{j=0}^{m} \binom{m}{j} q^{n(m-j)} (\Delta_q^{n+j} f)(0).
\]
Equating these formulas for $(\Delta_q^n f)(m)$ and isolating the $j = m$ term,
\[
c_{n+m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k + m) - \sum_{j=0}^{m-1} \binom{m}{j} q^{m-j} c_{n+j}.
\]
With this formula we show $|c_n| \to 0$.

The $j = 0$ term is $q^{m} c_n = q^{m} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k)$, so
\[
c_{n+m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k + m) - q^{m} f(k) - \sum_{j=1}^{m-1} \binom{m}{j} q^{m-j} c_{n+j}.
\]
Scaling, we may assume $|f(x)| \leq 1$ for all $x \in \mathbb{Z}_p$, so $|c_n| \leq 1$ for all $n$.

Let $m = p^r$, for $r$ to be determined. Then
\[
|c_{n+p^r}| \leq \max_{0 \leq k \leq n, 1 \leq j \leq p^r - 1} \left\{ |f(k + p^r) - q^{p^r} f(k)| \left| \binom{p^r}{j} q^{m-j} c_{n+j} \right| \right\}.
\]
For such $j$, $\left| \binom{p^r}{j} \right| q^j \leq |\Phi_{p^r}(q)|$ by (2.7).

Choose $\varepsilon > 0$. For large $r$, depending on $f$,
\[
|x - y| \leq \frac{1}{p^r} \Rightarrow |f(x) - f(y)| \leq \varepsilon.
\]
Thus $f(k + p^r) - q^{p^r} f(k) = f(k + p^r) - f(k) + f(k)(1 - q^{p^r})$, where the first term has size at most $\varepsilon$, while the second is at most $|q - 1| \max(|q - 1|, 1/p)^r$, which is $\leq \varepsilon$ for $r$ large (depending on $q$).

By Lemma 3.4, $|\Phi_{p^r}(q)| = 1/p$ for all large $r$, depending on $q$. So there is a large $r$ such that for all $n \geq 0$,
\[
|c_{n+p^r}| \leq \max_{1 \leq j \leq p^r - 1} (\varepsilon, (1/p)|c_{n+j}|) \leq \max(\varepsilon, 1/p).
\]
Thus $|c_n| \leq \max(\varepsilon, 1/p)$ for $n \geq p^r$. Replacing $n$ by $n + p^r$ gives, for all $n \geq 0$,
\[
|c_{n+p^r}| \leq \max_{1 \leq j \leq p^r - 1} (\varepsilon, (1/p)|c_{n+p^r+j}|) \leq \max(\varepsilon, 1/p^2).
\]
So
\[
|c_n| \leq \max(\varepsilon, 1/p^2)
\]
for $n \geq 2p^r$. Repeating this $s - 1$ more times gives
\[
|c_n| \leq \max(\varepsilon, 1/p^s)
\]
for $n \geq sp^r$. Choosing $s$ so large that $1/p^s \leq \varepsilon$ we have $|c_n| \leq \varepsilon$ if $n \geq sp^r$. \hfill \Box

Since the functions $E^x f$ are equicontinuous, this proof shows $\lim_{n \to \infty} \Delta^n_q f = 0$ uniformly in $q$ for $|q - 1| \leq \delta < 1$.

For the reader who knows about $q$-derivatives, a second way to obtain the two formulas for $(\Delta^n_q f)(m)$ in the proof above is to make the proposed equality of these two expressions a universal polynomial identity, and establish it by $q$-differentiating the equation

$$\sum_{k \geq 0} f(k) \frac{X^k}{(k)_q!} = E_q(X) \sum_{n \geq 0} c_n \frac{X^n}{(n)_q!},$$

$m$ times, dividing by $E_q(X)$, and then equating coefficients of $X^n$.

Although the $q$-Mahler expansion is treated above for a single function $f \in C(\mathbb{Z}_p, K)$, we will look in Section 5 at an example of a family of functions $f_q \in C(\mathbb{Z}_p, K)$ that depends continuously on $q$ and consider the expansion of $f_q$ relative to the $q$-Mahler basis.

4. Properties of $q$-Mahler Expansions

We now go through properties of $q$-Mahler expansions that are analogous to properties of Mahler expansions. Throughout this section, $|q - 1| < 1$.

First, note that $\{q \in K : |q - 1| < 1\}$ is a multiplicative group, unlike the parameter set that arises for $q$-series over $C$, the open unit disk. So we can also consider $1/q$-Mahler expansions.

**Theorem 4.1.** Let $|q - 1| < 1$, $f \in C(\mathbb{Z}_p, K)$ with $q$-Mahler coefficients $c_{n,q}$. Then

(i) $\sup_{x \in \mathbb{Z}_p} |f(x)| = \max_{n \geq 0} |c_{n,q}|$.

(ii) $f(x + 1) = \sum_{n \geq 0} (q^n c_{n,q} + c_{n+1,q}) (\frac{x}{q})^n q^n$.

(iii) $f(x + y) = \sum_{n \geq 0} (\Delta^n_q f)(y) (\frac{x}{n})_q^n q^n$.

(iv) $f(-x) = \sum_{n \geq 0} c_{n,q} (-1)^n q^{n(n+1)/2} (\frac{x}{n})_q^n$.

(v) $(x)_q f(x) = \sum_{n \geq 1} (n)_q (c_{n,q} + q^{n-1} c_{n-1,q}) (\frac{x}{n})_q^n$.

**Proof.** Part (i) follows from the $q$-Mahler Inversion Formula, or from the first proof of Theorem 3.3.

Part (ii) is a special case of part (iii) or can be done on its own. For part (iii), note $f(x + y) = (E^y f)(x)$ and the $n$th $q$-Mahler coefficient of $E^y f$ is $(\Delta^n_q (E^y f))(0) = (\Delta^n_q f)(y)$.

Part (iii) can also be proven by using the $q$-Vandermonde formula and an interchange of a double sum, which is Mahler’s original method at $q = 1$.

For part (iv), use (2.5). Note that the expansion given in (iv) is related to a $1/q$-Mahler expansion, which can be explicitly computed using Theorem 3.1.

For part (v), use $(n)_q (\frac{x}{n})_q = (x)_q (\frac{x}{n} - 1)_q$.

\hfill \Box

In light of (iii), $(\Delta^n_q f)(y)$ should be called the $n$th $q$-Mahler coefficient of $f$ at $y$.

As with Mahler expansions, a function $\mathbb{Z}_p \to K$ with a pointwise representation as $\sum c_{n,q} (\frac{x}{n})_q$ must be continuous, since $c_{n,q} \to 0$ by looking at $x = -1$.

Let’s see how the difference operators act on $q$-Mahler expansions. For $q = 1$,

$$\Delta^m \left( \sum_{j \geq 0} c_j (\frac{x}{j})_j \right) = \sum_{j \geq 0} c_{m+j} (\frac{x}{j})_j,$$
but for general \( q \) (2.10) implies

\[
\Delta^m_q \left( \sum_{j=0}^{\infty} c_{j,q} \left( \frac{x}{j} \right) \right) = \sum_{j=0}^{\infty} c_{m+j,q} q^{m(x-j)} \left( \frac{x}{j} \right),
\]

which is not a \( q \)-Mahler expansion, because of the term \( q^m \). So using the operator \( (\mathcal{D}_q^m f)(x) = q^{-m} (\Delta_q^m f)(x) \), we can write this instead as

\[
\mathcal{D}_q^m \left( \sum_{j=0}^{\infty} c_{j,q} \left( \frac{x}{j} \right) \right) = \sum_{j=0}^{\infty} c_{m+j,q} q^{-m} \left( \frac{x}{j} \right).
\]

The formula in part (v) of Theorem 4.1 can be extended to \( (x_m)_q f(x) \), computing the \( n \)th \( q \)-Mahler coefficient by (2.14) and (4.1) for \( n \geq m \):

\[
\Delta^m_q \left( \left( \frac{x}{m} \right)_q f(x) \right) (0) = \sum_{k=0}^{n} \binom{n}{k} q^{\Delta^k_q \left( \left( \frac{x}{m} \right)_q \right) (0)} \Delta^{n-k}_q (f)(k)
\]

\[
= \binom{n}{m} q^{\Delta^{n-m}_q (f)(m)}
\]

\[
= \binom{n}{m} q^{(n-m)k} \left( \frac{m}{k} \right) c_{n-k,q}.
\]

We now discuss the relation between differentiability and \( q \)-Mahler expansions. When \( q = 1 \), Mahler shows in [18, Theorem 3], [19] that \( f \in C(\mathbb{Z}_p, K) \) is differentiable at \( y \) if and only if \( \lim_{m \to \infty} (\Delta^m f)(y)/m = 0 \) and then

\[
f'(y) = \sum_{m \geq 1} \frac{(\Delta^m f)(y)}{m} (-1)^{m-1}.
\]

The extension of this result to general \( |q - 1| < 1 \) involves the \( p \)-adic logarithm, whose properties we will summarize for the convenience of readers outside of number theory. These readers should notice in particular part (iv) below, which says the \( p \)-adic logarithm is locally an isometry.

**Lemma 4.1.** (i) The series \( \log_p(1+z) = \sum_{n \geq 1} (-1)^{n-1} z^n / n \) converges at \( z \in K \) if and only if \( |z| < 1 \).

(iii) If \( |u_1 - 1|, |u_2 - 1| < 1 \), then \( \log_p(u_1 u_2) = \log_p(u_1) + \log_p(u_2) \).

(iv) For \( |q - 1| < 1 \), \( \lim_{x \to q} \frac{q^x - 1}{x} = \log_p q \).

(v) If \( |u - v| < (1/p)^{1/(p-1)} \), then \( |\log_p u - \log_p v| = |u - v| \).

(vi) \( \log_p u = 0 \) if and only if \( u \) is a \( p \)-th power root of unity in \( K \).

(vii) If \( |\zeta - 1| < 1 \) and \( \zeta^m = 1 \), then \( \lim_{q \to \zeta} \frac{\log_p q}{\zeta} = \frac{1}{m} \).

**Proof.** i) \( |z|^n \leq |z^n/n| \leq n|z|^n \).

ii) See [12, Prop. 4.5.3].

iii) For \( x \neq 0 \), \( (q^x - 1)/x = \sum_{n \geq 1} ((q - 1)^n/n)(x-1) \) and \( (q - 1)^n/n \to 0 \) by (i).

iv) By (ii) we may take \( v = 1 \). The first term of the series for \( \log_p u \) is \( u - 1 \). For \( u \neq 1 \), all the remaining terms have size less than \( |u - 1| \) since for \( n \geq 2 \), the unique minimum of \( |n|^{1/(n-1)} = (1/p)^{\text{ord}(n)/(n-1)} \) occurs at \( n = p \).
v) For any integer \( r \), \( \log_p u = 0 \) if and only if \( \log_p (u^r) = 0 \). For \( r \) large, \(|u^r - 1| < (1/p)^{1/(p-1)} \), and by (iv) the only \( z \) with \(|z - 1| < (1/p)^{1/(p-1)} \) and \( \log_p z = 0 \) is \( z = 1 \).

vi) \( (\log_p q)/((q^m - 1) = \log_p (q^z/(z^m - 1)) \) and \( \lim_{u \to 1} (\log_p u)/(u^m - 1) = 1/m \) since \( \lim_{u \to 1} (\log_p u)/(u - 1) = 1 \) from the definition of \( \log_p u \).

\[ \square \]

**Lemma 4.2.** Let \( g : \mathbb{Z}_p \to K \) be continuous on \( \mathbb{Z}_p - \{-1\} \), with \( g(x) = \sum_{n \geq 0} c_n(x)q \) for \( x \neq -1 \). Then \( g \) is continuous at \(-1\) if and only if \( c_n \to 0 \), in which case \( g(-1) = \sum_{n \geq 0} c_n(-1)q \).

**Proof.** The “if” direction is clear. For “only if”, continuity of \( g \) at \(-1\) is the same as continuity of \( g \) on \( \mathbb{Z}_p \), by our hypothesis. Letting \( x \) run through the nonnegative integers, we see by the \( q \)-Mahler Inversion Formula that \( c_n \) is the \( n \)th \( q \)-Mahler coefficient of \( g \), so we're done by Theorem 3.3. \[ \square \]

Here is the test for differentiability with \( q \)-Mahler expansions. Compare with formulas for the derivative in (4.2).

**Theorem 4.2.** Let \( f \in C(\mathbb{Z}_p, K) \).

(i) When \( q \) is not a (nontrivial) root of unity, \( f \) is differentiable at \( x \in \mathbb{Z}_p \) if and only if \( \lim_{m \to \infty} (\Delta_q^m f)(x)/(m)_q = 0 \), in which case

\[
f'(x) = \frac{\log_p q}{q - 1} \sum_{m \geq 1} (\Delta_q^m f)(x)(-1)^{m-1}q^{-m(m-1)/2}.
\]

(ii) When \( q \) is a root of unity of order \( p^N \) \((N \geq 0)\), \( f \) is differentiable at \( x \in \mathbb{Z}_p \) if and only if \( \lim_{m \to \infty} (\Delta_q^{p^N} f)(x)/(m)_q = 0 \), in which case

\[
f'(x) = \sum_{l \geq 1} \frac{\Delta_q^{p^N l} f)(x)}{p^N l}(-1)^{l-1}.
\]

**Proof.** (i) For \( h \neq 0 \), \( f(x + h) - f(x) = \sum_{m \geq 1} (\Delta_q^m f)(x) \binom{h}{m}_q \) by Theorem 4.1. Therefore

\[
\frac{f(x + h) - f(x)}{h} = \sum_{m \geq 1} (\Delta_q^m f)(x) \frac{1}{h} \binom{h}{m}_q
\]

\[
= \frac{(h)_q}{h} \sum_{m \geq 1} \frac{(\Delta_q^m f)(x)}{(m)_q} \binom{h - 1}{m - 1}.
\]

Since \((h)_q/h = (q^h - 1)/(h(q - 1))\) is continuous at all \( h \in \mathbb{Z}_p - \{0\}\) and its limit as \( h \to 0 \) is \( \frac{\log_q q}{q - 1} \neq 0 \) (even if \( q = 1 \), the function \( h/q \) is continuous and nowhere vanishing. So by Lemma 4.2 (with \( -1 \) as the variable), \( f'(x) \) exists if and only if \( (\Delta_q^m f)(x)/(m)_q \to 0 \) and then \( f'(x) \) has the indicated form.

(ii) We consider only suitably small \( h \), say \( h = p^N z \) for \( z \in \mathbb{Z}_p \). For \( z \neq 0 \),

\[
\frac{f(x + p^N z) - f(x)}{p^N z} = \sum_{m \geq 1} (\Delta_q^m f)(x) \frac{1}{p^N z} \binom{p^N z}{m}_q,
\]

and by (3.1),

\[
\binom{p^N z}{m}_q = \begin{cases} 
  \binom{z}{m/p^N}, & \text{if } p^N | m; \\
  0, & \text{if } p^N \not| m,
\end{cases}
\]

\[ ^{22} \]
\[
\frac{f(x + p^N z) - f(x)}{p^N z} = \sum_{l \geq 1} (\Delta_{q}^{p^N} f)(x) \frac{1}{p^N z} \left( \frac{z}{l} \right)
\]

\[
= \sum_{l \geq 1} \left( \frac{\Delta_{q}^{p^N} f(x)}{p^N l} \right) \left( \frac{z - 1}{l - 1} \right).
\]

Apply Lemma 4.2 (for \( q = 1 \)) with \( z - 1 \) as the variable.

Let’s unify both parts of this theorem. For \( q \) not a root of unity, \( \frac{\log_p q}{q-1} \frac{1}{(m^q)_{q^m}} = \frac{\log_p q}{q-1} \), while Lemma 4.1(vi) shows that for \( q \) a root of unity, \( \frac{(\log_p q)}{(q^m - 1)} \) equals \( 1/m \) when \( q^m = 1 \) and equals 0 otherwise. Moreover, if \( q \) is a root of unity of order \( p^N \), then \( \left( \frac{1}{p^N} \right)^{-1} = (-1)^{l-1} \) for \( l \geq 1 \). So for any \( |q - 1| < 1 \), a root of unity or not, \( f \) is differentiable at \( x \) if and only if \( \lim_{m \to \infty} (\Delta_{q}^{m} f)(x) \frac{\log_p q}{q^m - 1} = 0 \), in which case

\[
f'(x) = \sum_{m \geq 1} (\Delta_{q}^{m} f) \left( x \right) \frac{\log_p q}{q^m - 1} \left( \frac{-1}{m - 1} \right).
\]

In particular,

\[
f'(0) = \sum_{m \geq 1} c_m q \left( \frac{\log_p q}{q^m - 1} \right) \left( \frac{-1}{m - 1} \right).
\]

When \( f(x) = \sum c_n \binom{x}{n} \) is differentiable and \( f' \) is continuous, Mahler [18, Theorem 4] gives the Mahler expansion for \( f' \):

\[
(4.3) \quad f'(x) = \sum_{n \geq 0} \left( \sum_{j \geq 1} c_{n+j} (-1)^{j-1} \right) \binom{x}{n}.
\]

For the \( q \)-analogues, we use the following \( q \)-analogue of [22, Prop. 47.4]:

\[
p^k \leq n < p^{k+1} \Rightarrow \left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq p^k |x - y|.
\]

**Lemma 4.3.** Let \( n \geq 1 \), \( p^k \neq n < p^{k+1} \).

(i) When \( q \) is not a (nontrivial) root of unity,

\[
\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \frac{1}{[p^{k}]_q} |(x)_q - (y)_q| \leq \frac{1}{[p^{k}]_q} \max(|x - 1|, 1/p)^{\text{ord}(x - y)}.
\]

(ii) When \( q \) is a root of unity of order \( p^N \) (\( N \geq 0 \)) and \( x \equiv y \mod p^N \),

\[
\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq p^k |x - y|.
\]

**Proof.** i) Let \( x = y + z \), so by Theorem 3.1,

\[
\binom{x}{n}_q - \binom{y}{n}_q = \sum_{j=1}^{n} \binom{z}{j}_q \binom{z - 1}{j - 1}_q \binom{y}{n - j}_q q^{j(y + j - n)}.
\]
hence
\[ \left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \max_{1 \leq j \leq n} \left| \binom{z}{j}_q \right| = \max_{m \leq k} \frac{1}{(p^m)_q!} \left| (x)_q - (y)_q \right|. \]

ii) The difference vanishes if \( n < p^N \), so we may assume \( n \geq p^N \), i.e., \( k \geq N \). Let \( x \equiv y \equiv r \mod p^N \), \( 0 \leq r \leq p^N - 1 \). Write \( x = p^N x' + r, y = p^N y' + r, n = p^N l + s, 0 \leq s \leq p^N - 1 \), so \( p^{k-N} \leq l < p^{k+1-N} \). Then \( \binom{x}{n}_q - \binom{y}{n}_q = \left( \binom{x'}{l}_q - \binom{y'}{l}_q \right) \left( \binom{r}{s}_q \right) \), so (knowing the case \( q = 1 \) already)
\[ \left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leq \left| \binom{x'}{l}_q - \binom{y'}{l}_q \right| \leq p^{k-N} |x' - y'| = p^k |x - y|. \]

\[ \square \]

If \( |q - 1| < (1/p)^{1/(p-1)} \), then part (i) reduces to \( |\binom{x}{n}_q - \binom{y}{n}_q| \leq p^k |x - y| \), which (for \( q \in \mathbb{Z} \)) is a special case of [8, Theorem 4.5].

Here is the \( q \)-analogue of the Mahler expansion of \( f' \) when \( f' \) is continuous, extending (4.3).

**Theorem 4.3.** Let \( f(x) = \sum_{n \geq 0} c_{n,q}(\binom{x}{n})_q \) be a continuous function from \( \mathbb{Z}_p \) to \( K \) with a continuous derivative. The \( q \)-Mahler expansion of \( f' \) is
\[
f'(x) = \sum_{n \geq 0} \left( n c_{n,q} \log_p q + \sum_{j \geq 1} c_{n+j,q} \log_p q \frac{q^{j-1} - (j-1)_q}{q^j - 1} \binom{x}{n}_q \right).
\]

**Proof.** Apply \( \lim_{m \to \infty} (\Delta^m_q f)(x) \frac{\log q}{q^m - 1} = 0 \) at \( x = 0, 1, 2, \ldots \) to see \( \lim_{m \to \infty} c_{n+m,q} \frac{\log q}{q^m - 1} = 0 \) for all \( n \in \mathbb{N} \).

For \( y \neq 0 \),
\[
\frac{f(x + y) - f(x)}{y} = \sum_{n \geq 0} \left( \frac{\Delta^n_q f(y) - c_{n,q}}{y} \right) \binom{x}{n}_q.
\]

By (4.1),
\[
\frac{\Delta^n_q f(y) - c_{n,q}}{y} = c_{n,q} \left( \frac{q^m - 1}{y} \right) + \sum_{j \geq 1} c_{n+j,q} q^n (y-j) \frac{1}{y} \binom{j}{q}_q.
\]

How does each term behave as \( y \to 0 \)? The first term tends to \( c_{n,q} \log_p (q^n) = n c_{n,q} \log_p q \).

For the other terms,
\[
q^n (y-j) \frac{1}{y} \binom{j}{q}_q = q^n (y-j) \frac{q^y - 1}{y} \frac{1}{q^j - 1} \binom{y-1}{j-1}_q
\]
\[
\to \frac{\log_p q}{q^j - 1} \binom{y-1}{j-1}_q\]
\[
= \frac{\log_p q}{q^j - 1} \binom{y-1}{j-1}_q \binom{j}{q^j} = \binom{j}{q^j} = \left( \frac{q^j - 1}{q^j - 1} \right)_q.
\]
This calculation is valid only if \( q^j \neq 1 \), but the result is true if \( q^j = 1 \) by using (3.1). So we expect

\[
(4.4) \quad f'(x) = \sum_{n \geq 0} \left( n c_{n,q} \log_p q + \sum_{j \geq 1} c_{n+j,q} \log_p q \left( \frac{q^j}{q^j - 1} \right)^j \right) \left( \frac{x}{n} \right)_q.
\]

However, though we know \( \lim_{j \to \infty} c_{n+j,q} \log_p q \frac{q^j}{q^j - 1} = 0 \) for each \( n \), so the putative \( q \)-Mahler coefficients of \( f' \) in (4.4) do make sense, we don’t yet know

\[
\lim_{n \to \infty} \sum_{j \geq 1} c_{n+j,q} \log_p q \left( \frac{q^j}{q^j - 1} \right) \left( \frac{q^{-jn}}{q} \right)_q = 0,
\]

so convergence of the infinite series over \( n \) in (4.4) is not clear. To get around this, we use the idea of Mahler from his proof of Theorem 4.3 at \( q = 1 \), namely by the hypothesis of continuity of \( f' \) it suffices to verify (4.4) when \( x = m \in \mathbb{N} \). In this case the sum over \( n \) becomes finite:

\[
\frac{f(m+y) - f(m)}{y} = \sum_{n=0}^{m} \left( c_{n,q} \left( \frac{q^n - 1}{y} \right) + \sum_{j \geq 1} c_{n+j,q} q^{(n-j)} \left( \frac{y}{y} \right)_q \left( \frac{j}{j} \right)_q \right) \left( \frac{m}{n} \right)_q.
\]

The outer sum is finite, so to verify termwise evaluation of \( \lim_{y \to 0} \) all we need to do is check

\[
\lim_{y \to 0} c_{n+j,q} \frac{1}{y} \left( \frac{y}{y} \right)_q = c_{n+j,q} \log_p q \left( \frac{-1}{q^{j-1}} \right)_q
\]

uniformly in \( j \) (but perhaps not in \( q \) or \( n \)).

Case 1: \( q \) is a root of unity of order \( p^N \), so \( \lim_{j \to \infty} c_{n+j,q} / j = 0 \), as \( j \) runs through multiples of \( p^N \).

If \( q^j \neq 1 \), then \( \left( \frac{y}{y} \right)_q = 0 \) for \( |y| \leq 1/p^N \).

If \( q^j = 1 \), say \( j = p^N j' \), then

\[
\lim_{y \to 0} c_{n+j,q} \frac{1}{y} \left( \frac{y}{y} \right)_q = \lim_{z \to 0} c_{n+j,q} \frac{1}{j'} \left( \frac{z-1}{j'-1} \right)
\]

We consider the difference

\[
c_{n+j,q} \frac{1}{j'} \left( \frac{z-1}{j'-1} \right) - c_{n+j,q} \frac{1}{j} \left( \frac{-1}{j-1} \right)_q = \frac{c_{n+j,q}}{j} \left( \left( \frac{z-1}{j'-1} \right) - \left( \frac{-1}{j-1} \right) \right).
\]

Choose a power of \( p \), say \( p^r \), such that \( |c_{n+j,q} / j| \leq \delta \) for \( j \geq p^r \) (and \( p^N |j| \)). For \( j < p^r \), \((z^{-1}) - (j^{-1})\) has size at most \( p^{-1} |z| \) by Lemma 4.3.

Therefore

\[
\lim_{y \to 0} c_{n+j,q} \frac{1}{y} \left( \frac{y}{y} \right)_q = c_{n+j,q} \log_p q \left( \frac{-1}{q^{j-1}} \right)_q,
\]

uniformly in \( j \).

Case 2: \( q \) is not a root of unity.

So \( \log_p q \neq 0 \), hence \( \lim_{j \to \infty} \frac{c_{n+j,q}}{q^j-1} = 0 \).
Since
\[
c_{n+j,q} \frac{1}{y} \left( \frac{y}{j} \right)_q - c_{n+j,q} \frac{\log_p q}{q^j - 1} \left( \frac{y}{j} - 1 \right)_q = c_{n+j,q} \left( \frac{q^j - 1}{y} \left( \frac{y}{j} - 1 \right)_q - \log_p q \left( \frac{y}{j} - 1 \right)_q \right)
\]

\[
= \frac{c_{n+j,q}}{q^j - 1} \left( \frac{q^j - 1}{y} - \log_p q \right) \left( \frac{y}{j} - 1 \right)_q + \frac{c_{n+j,q} \log_p q}{q^j - 1} \left( \left( \frac{y}{j} - 1 \right)_q - \left( \frac{-1}{j} \right)_q \right),
\]

we need to show that
\[
\lim_{y \to 0} \frac{c_{n+j,q}}{q^j - 1} \left( \left( \frac{y}{j} - 1 \right)_q - \left( \frac{-1}{j} \right)_q \right) = 0
\]
uniformly in \( j \). For \( \delta > 0 \), choose \( p^r \) so \( |c_{n+j,q}/(q^j - 1)| \leq \delta \) for \( j \geq p^r \). For \( j < p^r \), Lemma 4.3 implies
\[
\left| \left( \frac{y}{j} - 1 \right)_q - \left( \frac{-1}{j} \right)_q \right| \leq \frac{1}{(p^r-1)^2} \max(|q - 1|, 1/p)^{\text{ord}(y)},
\]
which is \( \leq \delta \) for \( \text{ord}(y) \) large enough.

So for \( f' \) continuous and \( q \) not a root of unity,
\[
f'(x) = \log_p q \left( \sum_{n \geq 0} \left( (q-1)nc_n,q + \sum_{j \geq 1} c_{n+j,q}(-1)^{j-1}q^{-j(j-1)/2-jn} \right) \right) \left( \frac{x}{n} \right)_q,
\]
while for \( q \) a root of unity of order \( p^N \),
\[
f'(x) = \sum_{n \geq 0} \left( \sum_{j \geq 1 \mod p^N} c_{n+j,q}(-1)^{j/p^N-1} \right) \left( \frac{x}{n} \right)_q.
\]

The Mahler expansion characterizes analyticity: \( \sum c_n(x) \) is analytic if and only if \( c_n/n! \to 0 \) [22, Theorem 54.4]. For example, the function \( q^x \) is an analytic function of \( x \) if and only if \( |q - 1| < (1/p)^{1/(p-1)} \), in which case its \( m \)th Taylor coefficient at \( x = 0 \) is \( (\log_p q)^m / m! \). For other \( q \), \( |q^p - 1| < (1/p)^{1/(p-1)} \) for \( r \) large, so \( (x)_q \) is locally analytic.

To describe analyticity in terms of \( q \)-Mahler expansions, we only consider \( |q - 1| < (1/p)^{1/(p-1)} \), since this is the region of \( q \) where the functions \( (x)_q \) are all analytic. For such \( q \), \( |(x)_q| = |x| \). In particular, \( |n!| = |(n)_q!| \).

**Lemma 4.4.** Let \( a_1, b_1, \ldots, a_m, b_m \in K \) with \( |a_j|, |b_j| \leq 1 \). Then
\[
|a_1a_2 \cdots a_n - b_1b_2 \cdots b_n| \leq \max |a_j - b_j|.
\]

**Proof.** In \( \mathcal{O}_K/(a_1 - b_1, \ldots, a_n - b_n) \), \( a_1 \cdots a_n \equiv b_1 \cdots b_n \).

**Theorem 4.4.** For \( |q - 1| < (1/p)^{1/(p-1)} \), \( \sum c_n(x)_q \) is analytic if and only if \( c_n/(n)_q \to 0 \).

**Proof.** As with the first proof of Theorem 3.3, we’ll get the result for general \( q \) from the case \( q = 1 \) by Lemma 3.2.
Let \( A(\mathbb{Z}_p, K) = \{ f(x) = \sum a_n x^n : a_n \in K, a_n \to 0 \} \) be the analytic functions from \( \mathbb{Z}_p \) to \( K \). It is a \( K \)-Banach space under the norm \( ||f|| = \max |a_n| \). (This norm does not generally coincide with the sup-norm over \( \mathbb{Z}_p \), e.g., \( ||x^p - x|| = 1 \), but \( ||x^p - x||_{\text{sup}} = 1/p \).)

Writing
\[
\sum a_n x^n = \sum b_n x(x-1) \cdots (x-n+1) = \sum n!b_n \binom{x^n}{n},
\]
we see \( a_n - b_n \in \mathbb{Z}[[b_{n+1}, b_{n+2}, \ldots]] \), so \( \max |a_n| = \max |b_n| \). Therefore the norm in \( A(\mathbb{Z}_p, K) \) of an analytic function written as \( \sum c_n \binom{x^n}{n} \) is \( \max |c_n/n!| \). In other words, the functions \( n!(\binom{x}{n}) = x(x-1) \cdots (x-n+1) \) are a basis of \( A(\mathbb{Z}_p, K) \).

The theorem amounts to showing the functions \( (n)_q \binom{x}{n} = (x)_q(x-1)_q \cdots (x-n+1)_q \) are an orthonormal basis of \( A(\mathbb{Z}_p, K) \). To show this we compare these functions to \( n!(\binom{x}{n}) \) in order to use Lemma 3.2. By Lemma 4.4, it suffices to find an \( \varepsilon < 1 \) such that \( ||(x-j)_q - (x-j)|| \leq \varepsilon \) for all \( j \in \mathbb{N} \). Well,

\[
(x-j)_q - (x-j) = \left( \frac{\log_p q}{q-1} - 1 \right) (x-j) + \frac{\log_p q}{q-1} \sum_{r \geq 2} \frac{\log_p q}{r!} (x-j)^r.
\]

We want a uniform upper bound < 1 on the Taylor coefficients. (The definition of the norm on \( A(\mathbb{Z}_p, K) \) is based on a Taylor expansion around 0, but recentering the series at \( j \) does not affect the maximum size of the Taylor coefficients.)

The coefficient of \( x-j \) on the right side of (4.5) is
\[
\frac{\log_p q}{q-1} - 1 = \sum_{n \geq 2} \frac{(q-1)^{n-1}}{n} (-1)^{n-1}.
\]

Note \( |(q-1)^{n-1}/n| \leq |(q-1)^{n-1}/n!| \). By Lemma 4.1(iv), the coefficients of the higher powers of \( x-j \) in (4.5) have size
\[
\left| \frac{\log_p q}{q-1} \binom{(\log_p q)^{r-1}}{r!} \right| = \left| \frac{(q-1)^{r-1}}{r!} \right|.
\]

So provided \( \sup_{r \geq 2} |(q-1)^{r-1}/r!| < 1 \), we’re done. Letting \( s_p(r) \) be the sum of the base \( p \) digits of \( r \),
\[
\left| \frac{(q-1)^{r-1}}{r!} \right| = |q-1|^{r-1}p^{(r-s_p(r))/(r-1)} \leq |q-1|^{r-1}p^{(r-1)/(r-1)} \leq |q-1|p^{1/(p-1)}.
\]

\( \square \)

**Corollary 4.1.** For \( |q-1| < (1/p)^{1/(p-1)} \) and \( ||t|| < 1 \), \( (1+t)^{x\bar{q}} \) is analytic on \( \mathbb{Z}_p \) if and only if \( ||t|| < (1/p)^{1/(p-1)} \).

We now connect the work here with that of van Hamme and Verdooldt. They consider the following. Let \( a, q \in \mathbb{Z}_p^* \), perhaps \( q \neq 1 \mod p \), and assume \( q \) is not a root of unity. Let \( V_q \) denote the closure of the set \( \{ aq^n \}_{n \geq 0} \) in \( \mathbb{Z}_p \). It is a compact subset of \( \mathbb{Z}_p \) and open since \( q \) is not a root of unity. As \( q \to 1 \), \( V_q \) “shrinks” to \( \{ a \} \). In [23], van Hamme proves every continuous function \( f : V_q \to \mathbb{Q}_p \) has the form
\[
f(x) = \sum_{n \geq 0} \frac{(D^n f)(a)}{(n)_q!} (x-a)^{(n)_q}
\]
for \( x \in V_q \), where \((D_q f)(x) := (f(qx) - f(x))/(qx - x)\) is the \( q \)-derivative, \( D_q^n \) its \( n \)th iterate. Note that the domain \( V_q \) of the function depends on \( q \) and \( a \). Having \( (n)_q! \) in the denominator of (4.6) keeps \( q \) away from roots of unity.

When \( q \in 1 + p\mathbb{Z}_p \) and is not a root of unity, (4.6) is essentially a \( q \)-Mahler expansion. Indeed, in this case the elements of \( V_q \) have the form \( x = aq^y \) for unique \( y \in \mathbb{Z}_p \), in which case

\[
\frac{(D_q^n f)(a)}{(n)_q!}(x - a)^{(n)_q} = \frac{(D_q^n f)(a)}{(n)_q!}(aq^y - a)^{(n)_q} = (D_q^n f)(a) \cdot a^n(q - 1)^n\frac{q^n y - q^{n-1}}{q} \cdots \frac{q^{n-1} - 1}{q - 1} = (D_q^n f)(a) \cdot a^n(q - 1)^n\frac{y}{n!}.
\]

This last expression has an alternate form by [23, Lemma 3]:

\[
(D_q^n f)(a) \cdot a^n(q - 1)^n\frac{y}{n!} = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \binom{n}{k} \binom{y}{n}.
\]

This goes back to Jackson [14, Eq. 12].

Letting \( g(y) = f(aq^y) \) be the pullback of \( f \) to a continuous function on \( \mathbb{Z}_p \), van Hamme's expansion (4.6) becomes

\[
g(y) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \binom{n}{k} \binom{y}{n-k} \right) f(aq^{n-k}),
\]

which is the \( q \)-Mahler expansion of \( g \). But \( q \)-Mahler expansions do allow \( q \) to be a root of unity, as well as to lie outside of \( \mathbb{Q}_p \), though subject to the restriction \( |q| < 1 \). In [6], a \( q \)-analogue of Mahler expansions will be described for \( q \in K, |q| = 1 \), that will reduce to van Hamme's expansion when \( q \in \mathbb{Z}_p^* \) and \( q \) is not a root of unity.

In [24, Theorem 3], van Hamme gives a remainder formula for the Mahler expansion. For a complete extension field \( K/\mathbb{Q}_p \) and a continuous function \( f : \mathbb{Z}_p \rightarrow K \) with Mahler coefficients \( c_n \),

\[
(4.7) \quad f(x) = c_0 + c_1 \left( \frac{x}{1} \right) + \cdots + c_n \left( \frac{x}{n} \right) + \Delta^{n+1} f \ast^{t} \left( \frac{\cdot}{n} \right),
\]

where \( \ast^{t} \) is a modified convolution of continuous functions that we now recall. For two continuous functions \( g \) and \( h \) from \( \mathbb{Z}_p \) to \( K \), let \( g \ast^{t} h : \mathbb{Z}_p \rightarrow K \) be the \( p \)-adic interpolation to \( \mathbb{Z}_p \) of the function \( \mathbb{N} \rightarrow K \) given by \( n \mapsto \sum_{k=0}^{n} g(k)h(n-k) \). (For a proof that this sequence interpolates, see [22, Exer. 34.E, Exer. 52.J] or [24, Lemma 1].) The operation \( \ast^{t} \) is an associative, commutative convolution on \( C(\mathbb{Z}_p, K) \) and \( |g \ast^{t} h|_{sup} \leq |g|_{sup} |h|_{sup} \). By definition, \( (g \ast^{t} h)(x) := (g \ast h)(x - 1) \). Since \( \Delta^{n+1} f \rightarrow 0 \) in \( C(\mathbb{Z}_p, K) \), (4.7) is a Mahler expansion with remainder.

Here is the \( q \)-Mahler expansion with remainder.

**Theorem 4.5.** Choose \( q \in K \) with \( |q - 1| < 1 \) and \( f \in C(\mathbb{Z}_p, K) \). Letting \( c_{0,q}, c_{1,q}, \ldots \) be the \( q \)-Mahler coefficients of \( f \),

\[
f(x) = c_{0,q} + c_{1,q} \left( \frac{x}{1} \right) + \cdots + c_{n,q} \left( \frac{x}{n} \right) + \Delta^{n+1} f \ast^{t} \left( \frac{\cdot}{n} \right),
\]

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Our proof below will be a translation of Verdooldt’s ideas in [25], where she proves a version of this expansion with remainder for functions on the sets $V_q$. To simplify the comparison with [25], we write the variable in $\mathbb{Z}_p$ as $y$.

For $y \in \mathbb{Z}_p$, set $\mathcal{U}_n(y) = q^{ny}$, so $\mathcal{U}_0(y) = \left(\begin{array}{c} y \\ 0 \end{array}\right)_q$. (The functions $\mathcal{U}_n = \mathcal{U}_{n,q}$ were already used in Section 3.)

**Lemma 4.5.** For any $n \geq 0$, $f = f(0)\mathcal{U}_n + (E - q^n)f *' \mathcal{U}_n$.

**Proof.** We evaluate the right hand side at $y = m \in \mathbb{Z}^+$:

\[
(E - q^n)f *' \mathcal{U}_n(m) = \sum_{i=0}^{m-1} (f(i + 1) - q^n f(i)) q^{n(m-1-i)}
\]

\[
= \sum_{i=0}^{m-1} f(i + 1)q^{n(m-(i+1))} - \sum_{i=0}^{m-1} f(i)q^{n(m-i)}
\]

\[
= f(m) - f(0)q^m.
\]

**Lemma 4.6.** For all $n$,

\[
\mathcal{U}_{n+1} *' \left(\begin{array}{c} m \\ n \end{array}\right)_q = \left(\begin{array}{c} m+1 \\ n+1 \end{array}\right)_q.
\]

**Proof.** Using the first recursion in (2.3),

\[
\left(\begin{array}{c} m \\ n+1 \end{array}\right)_q = \left(\begin{array}{c} m-1 \\ n \end{array}\right)_q + q^{n+1}\left(\begin{array}{c} m-1 \\ n+1 \end{array}\right)_q
\]

\[
= \left(\begin{array}{c} m-1 \\ n \end{array}\right)_q + q^{n+1}\left(\begin{array}{c} m-2 \\ n \end{array}\right)_q + q^{2(n+1)}\left(\begin{array}{c} m-2 \\ n+1 \end{array}\right)_q
\]

\[
= \sum_{i=0}^{m-1} \left(\begin{array}{c} m-1-i \\ n \end{array}\right)_q q^{i(n+1)}
\]

\[
= \mathcal{U}_{n+1}(y) * \left(\begin{array}{c} y \\ n \end{array}\right)_q \text{ at } y = m-1
\]

\[
= \mathcal{U}_{n+1}(y) *' \left(\begin{array}{c} y \\ n \end{array}\right)_q \text{ at } y = m.
\]

Now we prove Theorem 4.5.

**Proof.** For $y \in \mathbb{Z}_p$,

\[
f(y) = f(0)\mathcal{U}_0 + (E - I)f *' \mathcal{U}_0
\]

\[
= f(0) + \Delta f *' \mathcal{U}_0
\]

\[
= f(0) + ((\Delta f)(0)\mathcal{U}_1 + (E - q)\Delta f *' \mathcal{U}_1) *' \mathcal{U}_0 \text{ by Lemma 4.5}
\]

\[
= f(0) + (\Delta f)(0)(\mathcal{U}_1 *' \mathcal{U}_0) + \Delta^2 f *' (\mathcal{U}_1 *' \mathcal{U}_0)
\]

\[
= f(0) + (\Delta f)(0)\left(\begin{array}{c} y \\ 1 \end{array}\right)_q + \Delta^2 f *' \left(\begin{array}{c} y \\ 1 \end{array}\right)_q \text{ by Lemma 4.6}.
\]
Assuming
\[ f(y) = f(0) + (\Delta_q f)(0) \left( \frac{y}{1} \right)_q + \cdots + (\Delta_q^n f)(0) \left( \frac{y}{n} \right)_q + \Delta_q^{n+1} f \circ \left( \frac{\cdot}{n} \right)_q, \]
apply Lemma 4.5 at \( n + 1 \) with the function \( \Delta_q^{n+1} f \), and then use Lemma 4.6. \( \square \)

It is left to the reader to extend the \( q \)-Mahler expansion and some properties of it in this section to the case when \( K \) is a complete field of characteristic \( p \) or a complete commutative \( \mathbb{Z}_p \)-algebra.

In addition to the \( q \)-numbers and \( q \)-binomial coefficients we have used, the study of quantum groups has focused attention on the \( q \)-analogues
\[ [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + \frac{1}{q^{n-3}} + \frac{1}{q^{n-1}} = \frac{1}{q^{n-1}} (n)_q^2, \]
\[ [n]_q! := [n]_q [n-1]_q \cdots [1]_q = \frac{1}{q^{n(n-1)/2}} (n)_q^2!, \]
and
\[ \left[ \begin{array}{c} m \\ n \end{array} \right]_q := \left[ \begin{array}{c} m \\ n \end{array} \right] [m-1]_q \cdots [m-n+1]_q = \frac{1}{q^{m-n}} \left( \begin{array}{c} m \\ n \end{array} \right)_q. \]
The extra property these have is invariance when \( q \) is replaced by \( 1/q \).

All the properties of \( \binom{m}{n} \) have analogues for \( \left[ \binom{m}{n} \right]_q \), such as
\[ \left[ -n \right]_q = -[n]_q, \quad [n]_{1/q} = [n]_q, \quad [mn]_q = [m]_q [n]_q^m, \quad \left[ \begin{array}{c} m \\ n \end{array} \right]_q \in \mathbb{Z}[q, 1/q], \]
\[ \left[ \begin{array}{c} -m \\ n \end{array} \right]_q = (-1)^n \left[ \begin{array}{c} m + 1 - n \\ n \end{array} \right]_q, \quad \left[ \begin{array}{c} m_1 + m_2 \\ k \end{array} \right]_q = \sum_{i+j=k} \left[ \begin{array}{c} m_1 \\ i \end{array} \right]_q \left[ \begin{array}{c} m_2 \\ j \end{array} \right]_q q^{m_2 i - m_1 j}. \]
That \( \left[ \binom{m}{n} \right]_q \) is related to \( \binom{m}{n} \) means there is a different formula for \( \left[ \binom{m}{n} \right]_q \) in the case when \( \zeta \) is an odd or even order root of unity.

For \( |q - 1| < 1 \), we get a continuous extension \( \left[ x \right]_q = (1/q^n \frac{x}{n}) \left( \frac{x}{n} \right)_q^2 \), and \( \left[ \frac{x}{n} \right]_q - \left( \frac{x}{n} \right) \) is \( |q - 1| \), so the functions \( \left[ x \right]_q \) form an orthonormal basis of \( C(\mathbb{Z}_p, K) \).

It is left to the reader to formulate all the results of this paper so far in this context. As an example of some differences, let \( \mathcal{E}_q(X) = \sum X^n/[n]_q! \). Then \( \mathcal{E}_{1/q}(X) = \mathcal{E}_q(X) \) and \( \mathcal{E}_q(X) \mathcal{E}_q(Y) \) equals
\[ \sum_{n \geq 0} \frac{1}{[n]_q!} \left( \sum_{m=0}^n \left[ \begin{array}{c} n \\ m \end{array} \right]_q X^{n-m} Y^m \right) = \sum_{n \geq 0} \left( \frac{X + Y/q^{n-1}}{[n]_q!} \right) \left( X + Y/q^{n-3} \right) \cdots \left( X + q^{n-1} Y \right), \]
where powers of \( q \) in consecutive terms of the product on the right hand side differ by two.

Set
\[ (X + Y)^{[n]}_q := (X + Y/q^{n-1}) \cdots (X + q^{n-1} Y), \]
so \( \mathcal{E}_q(X) \mathcal{E}_q(Y) = \sum_{n \geq 0} (X + Y)^{[n]}_q/[n]_q! \) and \( (X + Y)^{[m+n]} q^{[m]}_q q^{[n]}_q = (X + q^n Y)^{[m]}_q (X + Y/q^m)^{[n]}_q \).

Note \( (X - X)^{[n]}_q \neq 0 \) if \( n \) is even. In particular, \( \mathcal{E}_q(X) \mathcal{E}_q(-X) \neq 1 \), and there doesn’t seem to be a simple formula for the coefficients of \( \mathcal{E}_q(X)^{-1} \). For example,
\[ \mathcal{E}_q(X)^{-1} = 1 - X + \frac{q^2 - q + 1}{q^2 + 1} X^2 - \frac{q^6 - 2q^5 + 2q^4 - q^3 + 2q^2 - 2q + 1}{(1 + q^2)(1 + q^2 + q^4)} X^3 + \cdots, \]
and the numerator of the coefficient of $X^3$ is irreducible in $\mathbb{Z}[q]$.

We define polynomials $\mu_n(q)$ by $E_q(X) = \sum_{n \geq 0} \mu_n(q)X^n/[n]_q!$, using the notation $\mu$ by analogy with combinatorial inversion formulas. Then

$$f(x) = \sum_{n \geq 0} C_{n,q} \left[ \frac{x}{n} \right] \iff C_{n,q} = \sum_{k=0}^{n} \binom{n}{k} \mu_{n-k}(q) f(k).$$

5. The $p$-adic $q$-Gamma function

To illustrate the possibility of using $q$-Mahler expansions with a family of functions depending continuously on a parameter, we consider Morita’s $p$-adic Gamma function $\Gamma_p$ and its $q$-analogue $\Gamma_{p,q}$ as defined by Koblitz.

For a nonnegative integer $n$, Morita [20] defines

$$\Gamma_p(n + 1) := (-1)^{n+1} \prod_{\substack{1 \leq j \leq n \\ (p,j) = 1}} j = (-1)^{n+1} n! \frac{1}{p^{n/n!} [n/p]!}$$

for $n \geq 1$ and $\Gamma_p(1) = -1$. Morita’s proof that $\Gamma_p$ is $p$-adically continuous is based on congruence properties of the sequence $\{\Gamma_p(n+1)\}$. For our treatment here, it is Barsky’s proof [1] of the continuity which is of primary interest. Barsky’s method is based on the identity

$$\sum_{n \geq 0} \frac{(-1)^{n+1} \Gamma_p(n + 1)}{n!} X^n = (1 + X + \cdots + X^{p-1})e^{X/p},$$

which implies that the Mahler coefficients $\tau_p(n)$ (say) of the sequence $\Gamma_p(n + 1)$ satisfy

$$\sum_{n \geq 0} \frac{(-1)^{n+1} \tau_p(n)}{n!} X^n = (1 + X + \cdots + X^{p-1})e^{X+X/p}.$$}

Writing $e^{X+X/p} = \sum_{n \geq 0} (b_{p,n}/n!) X^n$, estimates of Dwork [17, p. 320] imply $b_{p,n} \to 0$ $p$-adically as $n \to \infty$, so $\tau_p(n) \to 0$ as $n \to \infty$. Therefore $\Gamma_p$ extends continuously from $\mathbb{N}$ to $\mathbb{Z}_p$.

We recall Dwork’s proof that $b_{p,n} \to 0$. Multiply $\exp(X + X^p/p)$ by the additional terms $\exp(X^{p^j/p^j})$ for $j \geq 2$ and then remove them:

$$e^{X+X^p/p} = \exp \left( \sum_{j \geq 0} \frac{X^{p^j}}{p^j} \right) \prod_{j \geq 2} e^{-X^{p^j/p^j}}.$$

We want to show $\exp(X + X^p/p)$ is in the space of $p$-adic divided power series $\sum c_n X^n/n!$ where $c_n \to 0$. Such series form the Leopoldt space. It is a Banach algebra when we norm such series by $\sup |c_n|$. Since $\exp(\sum_{j \geq 0} X^{p^j/p^j})$ is the Artin–Hasse series, which has $\mathbb{Z}_p$-coefficients, it is a Leopoldt series. (Any series with bounded coefficients is a Leopoldt series.) By a direct calculation, $\exp(\pm X^{p^j/p^j})$ is a Leopoldt series and $\to 1$ in the Leopoldt norm as $j \to \infty$. So by completeness the right side of (5.3) is a Leopoldt series. Thus $b_{p,n} \to 0$.

For $|q - 1| < 1$, the $q$-analogue $\Gamma_{p,q}$ of $\Gamma_p$ is defined by Koblitz [16] by

$$\Gamma_{p,q}(n + 1) := (-1)^{n+1} \prod_{\substack{1 \leq j \leq n \\ (p,j) = 1}} q^{j-1} - 1 = (-1)^{n+1} \prod_{\substack{1 \leq j \leq n \\ (p,j) = 1}} (1 + q + \cdots + q^{j-1})$$

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for \( n \geq 1 \) and \( \Gamma_{p,q}(1) = -1 \). For fixed \( q \) with \( 0 < |q - 1| < 1 \), Koblitz shows that the sequence \( \Gamma_{p,q}(n + 1) \) \( p \)-adically interpolates to \( \mathbb{Z}_p \) by comparing \( \Gamma_{p,q} \) with \( \Gamma_p \), whose continuity is already known. There are alternate proofs of the interpolation for \( \Gamma_{p,q} \) (cf. [5]), but we would like to have available a proof of the interpolation based on Barsky’s method, proceeding as follows.

For any integer \( j \), \( (j)_{q_2} \equiv (j)_{q_1} \mod q_1 - q_2 \), so \( |\Gamma_{p,q_1}(n + 1) - \Gamma_{p,q_2}(n + 1)| \leq |q_1 - q_2| \). Thus \( p \)-adic interpolation of \( \Gamma_{p,q}(n + 1) \) for general \( q \) will follow from that for a dense set of \( q \). So we may suppose \( q \) is not a root of unity, making \( (n)_q \) nonzero for all \( n \).

In this case, which we may assume we are in from now on,

\[
\Gamma_{p,q}(n + 1) = \frac{(-1)^{n+1} (n)_q}{\prod_{k \leq [n/p]} (pk)_q} = \frac{(-1)^{n+1} (n)_q}{(p)^{n/p}([n/p])!_q}.
\]

Following Barsky, we consider

\[
\sum_{n \geq 0} \frac{(-1)^{n+1} \Gamma_{p,q}(n + 1)}{(n)_q} X^n = \sum_{n \geq 0} \frac{1}{(p)^{n/p}([n/p])!_q} X^n
= \sum_{r=0}^{p-1} \sum_{m \geq 0} \frac{1}{(p)^{m/n}} X^{mn+r}
= (1 + X + \cdots + X^{p-1}) \sum_{m \geq 0} \frac{(X^{p/(p)_q})^m}{(m)_q!_q}
= (1 + X + \cdots + X^{p-1}) E_{q^p}(X^{p/(p)_q})
\]

Let \( \tau_{p,q}(n) \) be the \( n \)th \( q \)-Mahler coefficient of the sequence \( \Gamma_{p,q}(n + 1) \). We want to show \( \tau_{p,q}(n) \to 0 \) as \( n \to \infty \). Continuing with the above calculations, we obtain

\[
\sum_{n \geq 0} \frac{(-1)^{n+1} \tau_{p,q}(n)}{(n)_q} X^n = (1 + X + \cdots + X^{p-1}) E_{q^p}(-X)^{-1} E_{q^p}(X^{p/(p)_q})
= (1 + X + \cdots + X^{p-1}) E_{1/q}(X) E_{q^p}(X^{p/(p)_q}).
\]

Comparing this with (5.2) shows the \( q \)-analogue of \( e^{X+X^n/p} \) is apparently

\[
E_{1/q}(X) E_{q^p}(X^{p/(p)_q}) = (E_{1/q}(X) E_{1/q}(-X)) \cdot E_q(X) E_{q^p}(X^{p/(p)_q}).
\]

By the \( q \)-Mahler theorem, the existence of a \( p \)-adic interpolation for \( \Gamma_{p,q}(n + 1) \) is thus equivalent to the fact that, when we write

\[
E_{1/q}(X) E_{q^p}(X^{p/(p)_q}) = \sum_{n \geq 0} b_{p,q,n} \frac{X^n}{(n)_q!_q},
\]

the sequence \( b_{p,q,n} \) tends to 0 as \( n \to \infty \). This suggests looking at a \( q \)-Leopoldt space, namely the \( q \)-divided power series \( \sum c_n X^{n/(n)_q} \) where \( c_n \to 0 \). By a direct calculation for \( j \geq 2 \), \( E_{q^{p^j}}(X^{p^j/(p^j)_q}) \) is a unit in the \( q \)-Leopoldt space, so carrying out a \( q \)-version of Barsky’s argument comes down to checking that a \( q \)-analogue of the Artin–Hasse series,

\[
E_{1/q}(X) \prod_{j \geq 1} E_{q^{p^j}}(X^{p^j/(p^j)_q}),
\]

(5.4)
is in the $q$-Leopoldt space. (Since $E_{1/q}(X)E_{1/q}(-X)$ is a unit in the $q$-Leopoldt space, we can replace $E_{1/q}(X)$ with $E_q(X)$ in (5.4) without affecting the property of being or not being a $q$-Leopoldt series.)

Here we are left with a gap, as we do not see how to establish (5.4) is a $q$-Leopoldt series without referring to the preexisting fact that $\Gamma_{p,q}(n + 1)$ interpolates. Is there a method of analyzing (5.4) without using anything about $\Gamma_{p,q}$, and ideally also not relying on the case $q = 1$ first? It may be possible to carry out this task more easily when $|q - 1| < (1/p)^{1/(p-1)}$, but ultimately there should be an argument valid for $|q - 1| < 1$.

References