Show all work clearly. For full credit, include all steps necessary for your answer to be a clear and logical consequence of the preceding work. To be eligible for partial credit, all row-reduction work must carry clear labels indicating which operations you perform.

Budget time by allotting no more than 7 minutes for each 10 points.

1. Let $B = \begin{bmatrix} 1 & 2 & 2 \\ -6 & -11 & -12 \\ 1 & -4 & 3 \end{bmatrix}$.

(a) (10 points) Find the $LU$-decomposition of $B$.

(b) (5 points) Without further work, calculate $\det B$ and explain why $B$ is invertible.

(c) (5 points) Calculate $U^{-1}$.

(d) (5 points) Use your work to write $L$ as a product of elementary matrices.

(e) (5 points) Calculate $L^{-1}$ from (d).

(f) (5 points) Calculate $B^{-1}$ from (c) and (e). (This is the procedure MATLAB actually follows to invert matrices.)

(g) (5 points) Write $B$ as a product of elementary matrices. (Do not check by multiplying out the product!)

2. (15 points) Consider the linear transformation $T$ whose standard matrix is

$$C = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -4 & 3 \\ -6 & -11 & -12 \end{bmatrix}.$$

(a) What is the value of $\det C$? (b) Is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ in the range of $T$? Why or why not?

(c) Describe the image under $T$ of the unit cube determined by the vectors $i, j, k$ in $\mathbb{R}^3$.

3. (10 points) Suppose that $I, O, X, Y, A, B, C$, and $D$ have appropriate sizes so that

$$\begin{bmatrix} I & X \\ O & Y \end{bmatrix} \begin{bmatrix} A & C \\ B & O \end{bmatrix} = \begin{bmatrix} D & C \\ I & O \end{bmatrix}.$$ Express $X$ and $Y$ in terms of $A, B, C$, and $D$.

4. (15 points, 5 per part) Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be the bases of $\mathbb{R}^2$ for which $b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, and $c_2 = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$. 
(a) Find the coordinate vector \([v]_B\) of \(v = \begin{bmatrix} -3 \\ 7 \end{bmatrix}\) relative to the basis \(B\).

(b) Find the change-of-basis matrix \(P = P_{C \leftarrow B}\).

(b) Give the matrix-vector equation that connects \([v]_B\) and \([v]_C\). Use that to calculate \([v]_C\).

5. (20 points, 5 per part) Let \(A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ -1 & 3 & 5 \end{bmatrix}\).

(a) Find the rank of \(A\).

(b) What is the dimension of the column space \(R(A)\) of \(A\)? Give a basis for \(R(A)\).

(c) What is the dimension of the null space \(N(A)\) of \(A\)? Give a basis for \(N(A)\).

(d) What is the dimension of the column space of \(A^T\)? What is the dimension of the null space of \(A^T\)?

6. Bonus Question (10 points maximum) Attempt this only after having completed and checked the earlier problems.

Two square matrices \(A\) and \(B\) anticommute if \(AB = -BA\).

(a) Show that if two 3-by-3 matrices anticommute, then at least one of them is not invertible.

(b) What if \(A\) and \(B\) are 2-by-2 matrices?