1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ if $m, n$ are coprime.

Proof. Since $m$ and $n$ are coprime, then there exists some $s, t \in \mathbb{Z}$ such that

$$ms + nt = 1.$$ 

Now for any simple tensor $x \otimes y \in (\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$, we have

$$x \otimes y = 1 \cdot (x \otimes y)$$

$$= (ms + nt) \cdot (x \otimes y)$$

$$= (ms) \cdot (x \otimes y) + (nt) \cdot (x \otimes y)$$

$$= (msx) \otimes y + (ntx) \otimes y$$

$$= (msx) \otimes y + x \otimes (tnt)$$

$$= 0 \otimes y + x \otimes 0$$

$$= 0 + 0$$

$$= 0.$$ 

Since $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ is generated by simple tensors, then we have

$$(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0.$$

2. Let $A$ be a ring, $a$ an ideal, $M$ an $A$-module. Show that $(A/a) \otimes_A M$ is isomorphic to $M/aM$.

Proof. Define $f : A/a \times M \to M/aM$ as: for all $x + a \in A/a$ and $m \in M$, we have

$$f(x + a, m) = xm + aM.$$ 

Claim I: $f$ is well defined.

In fact, for all $x + a, y + a \in A/a$ and $m \in M$ such that $x + a = y + a$, then $x - y \in a$. Hence $xm = ym = (x - y)m \in aM$, in particular,

$$xm + aM = ym = aM.$$ 

That is, $f(x + a, m) = f(y + a, m)$. So $f$ is well defined.

Claim II: $f$ is an $A$-bilinear map.

In fact, for all $z \in A$, $x + a, y + a \in A/a$ and $m, n \in M$, then

$$f(z(x + a) + (y + a), m) = f(zx + y + a, m)$$

$$= (zx + y)m + aM$$

$$= zxm + ym + aM$$

1
\[ f(x + a, zm + n) = x(zm + n) + aM \]
\[ = zxm + xn + aM \]
\[ = (zxm + aM) + (xn + aM) \]
\[ = zf(x + a, m) + f(x + a, n). \]

Hence \( f \) is an \( A \)-bilinear map.

Since \( f \) is an \( A \)-bilinear map, by the universal property of tensor product, then there exists a unique \( A \)-module homomorphism \( \varphi : A/a \otimes_A M \to M/aM \) such that for all \( x \in A \) and \( m \in M \), we have
\[ \varphi((x + a) \otimes m) = xm + aM. \]

Define another map \( \psi : M/aM \to A/a \otimes_A M \) as: for all \( m + aM \), we have
\[ \psi(m + aM) = (1 + a) \otimes m. \]

Claim III: \( \psi \) is well defined.

In fact, for all \( m, n \in M \) such that \( m - n \in aM \), then there exists some \( a \in a \) and \( l \in M \) such that \( m - n = al \). Hence
\[ \psi(m + aM) = (1 + a) \otimes m \]
\[ = (1 + a) \otimes (al + n) \]
\[ = (1 + a) \otimes (al) + (1 + a) \otimes n \]
\[ = [a(1 + a)] \otimes l + (1 + a) \otimes n \]
\[ = (a + a) \otimes l + (1 + a) \otimes n \]
\[ = 0 \otimes l + (1 + a) \otimes n \]
\[ = 0 + (1 + a) \otimes n \]
\[ = (1 + a) \otimes n \]
\[ = \psi(n + aM). \]

Claim IV: \( \psi \) is an \( A \)-module homomorphism.

In fact, for all \( m, n \in M \) and \( x \in A \), we have
\[ \psi(x(m + aM) + n + aM) = f(xm + n + aM) \]
\[ = (1 + a) \otimes (xm + n) \]
\[ = (1 + a) \otimes (xm) + (1 + a) \otimes n \]
\[ = x(1 + a) \otimes m + (1 + a) \otimes n \]
\[ = x\psi(m + aM) + \psi(n + aM). \]

Hence \( \psi \) is an \( A \)-module homomorphism.

Claim V: \( \psi \circ \varphi = Id. \)
In fact for all simple tensor \((x + a) \otimes m \in A/a \otimes_A M\), then

\[
\psi \circ \varphi((x + a) \otimes m) = \psi(xm + aM) = (1 + a) \otimes (xm) = [x(1 + a)] \otimes m = (x + a) \otimes m.
\]

Since \(A/a \otimes_A M\) is generated by simple tensors, then \(\psi \circ \varphi = Id\) on \(A/a \otimes_A M\).

Claim VI: \(\varphi \circ \psi = Id\).

For all \(m + aM \in M/aM\), then

\[
\varphi \circ \psi(m + aM) = \varphi((1 + a) \otimes m) = 1m + aM = m + aM.
\]

Hence \(\varphi \circ \psi = Id\) on \(M/aM\).

In summary, we know that \(\varphi\) and \(\psi\) are \(A\)-module isomorphisms. Hence we know that \((A/a) \otimes_A M\) is isomorphic to \(M/aM\).

\[\square\]

3. Let \(A\) be a local ring, \(M\) and \(N\) finitely generated \(A\)-modules. Prove that if \(M \otimes_A N = 0\), then \(M = 0\) or \(N = 0\).

**Proof.** Since \(A\) is a local ring, then \(A\) has a unique maximal ideal \(m\) in \(A\). Since \(m\) is the unique maximal ideal in \(A\), then the Jacobson radical \(J\) of \(A\) is equal to \(m\) and \(k = A/m\) is a field.

For any \(A\)-module \(L\), let \(L_k = k \otimes_A L\). By the result of the Problem 2, then

\[L_k = k \otimes_A L = A/m \otimes_A L \cong L/mL.\]

Then \(L_k\) is a \(k\)-vector space. Since \(M \otimes_A N = 0\), then \((M \otimes_A N)_k = 0\). On the other hand, since \(k \otimes_k k = k\), then we know that

\[
(M \otimes_A N)_k = k \otimes_A (M \otimes_A N) = k \otimes_A M \otimes_A N = M \otimes_A k \otimes_A N = M \otimes_A (k \otimes_k k) \otimes_A N = (M \otimes_A k) \otimes_k (k \otimes_A N) = M_k \otimes_k N_k.
\]

Hence \(M_k \otimes_k N_k = 0\). Since \(M_k \otimes_k N_k\) is a \(k\)-vector space of dimension \(\dim M_k \cdot \dim N_k\), hence we must have \(M_k = 0\) or \(N_k = 0\). Without loss of generality, we assume \(M_k = 0 = k \otimes_A M \cong M/mM\). Hence we get

\(M = mM\).

Since \(J = m\) and \(M, N\) are finitely generated \(A\)-modules, by the Nakayama’s Lemma, we know that \(M = 0\).

\[\square\]

4. Let \(M_i (i \in I)\) be any family of \(A\)-modules, and let \(M\) be their direct sum. Prove that \(M\) is flat \(\iff\) each \(M_i\) is flat.
Proof. ($\Rightarrow$) Assume $M = \bigoplus_{i \in I} M_i$ is flat. For all $i \in I$, define $\pi_i : M \to M_i$ as the $i$-th projection, that is, for all $(m_j)_{j \in I} \in M$, we have

$$\pi((m_j)_{j \in I}) = m_i.$$ 

Let $e_i : M_i \to M$ as the $i$-th embedding, that is, for all $m_i \in M_i$, let $m_j = m_i$ if $i = j$ and $m_j = 0$ if $i \neq j$, then we have

$$e_i(m_i) = (m_j)_{j \in I}.$$ 

Now for any $A$-modules $N$ and $N'$ with any injective $A$-module homomorphism $f : N \to N'$. Since $M$ is flat, then 

$$f \otimes 1_M : N \otimes_A M \to N' \otimes_A M$$

is injective.

Let $\overline{f} : N \to f(N) \subset N'$, then $\overline{f}$ is bijective. Since $M$ is flat, then

$$\overline{f} \otimes 1_M : N \otimes_A M \to f(N) \otimes_A M$$

is injective.

Since

$$N \otimes_A M = N \otimes_A \left( \bigoplus_{i \in I} M_i \right) = \bigoplus_{i \in I} (N \otimes_A M_i),$$

and

$$N' \otimes_A M = N' \otimes_A \left( \bigoplus_{i \in I} M_i \right) = \bigoplus_{i \in I} (N' \otimes_A M_i),$$

Then

$$1_N \otimes e_i : N \otimes_A M_i \to N \otimes_A M$$

is injective.

So we get

$$(f \otimes 1_M) \circ (1_N \otimes e_i) : N \otimes_A M_i \to N' \otimes_A M$$

is injective.

Now for $f \otimes 1_{M_i} : N \otimes_A M_i \to N' \otimes_A M_i$, we want to show that $f \otimes 1_{M_i}$ is injective, since $f \otimes 1_{M_i}(N \otimes_A M_i) \subset f(N) \otimes_A M_i$, then it suffices to show $\overline{f} \otimes 1_{M_i} : N \otimes_A M_i \to f(N') \otimes_A M_i$ is injective. Since $1_{M_i} = \pi_i \circ 1_M \circ e_i$, then

$$\overline{f} \otimes 1_{M_i} = (1_{f(N)} \otimes \pi_i) \circ (\overline{f} \otimes 1_M) \circ (1_N \otimes e_i).$$

Hence $\overline{f} \otimes 1_{M_i}$ is injective. So we know that $f \otimes 1_{M_i} : N \otimes_A M_i \to N' \otimes_A M_i$ is injective. Therefore, we know that $M_i$ is flat for all $i \in I$.

($\Leftarrow$) Assume that for all $i \in I$, $M_i$ is flat. Now for any $A$-modules $N$ and $N'$ with any injective $A$-module homomorphism $f : N \to N'$. Since $M_i$ is flat, then

$$f \otimes 1_{M_i} : N \otimes_A M_i \to N' \otimes_A M_i$$

is injective.

Now consider $f \otimes 1_M : N \otimes_A M \to N' \otimes_A M$, for any $\left( \sum_{\text{finite}} n_j \otimes (m^j_i)_{i \in I} \right) \in \ker f \otimes 1_M$, that is,

$$f \otimes 1_M \left( \sum_{\text{finite}} n_j \otimes (m^j_i)_{i \in I} \right) = 0$$

Then

$$0 = f \otimes 1_M \left( \sum_{\text{finite}} n_j \otimes (m^j_i)_{i \in I} \right)$$

$$= \sum_{\text{finite}} f(n_j) \otimes (m^j_i)_{i \in I}$$

$$= \left( \sum_{\text{finite}} f(n_j) \otimes m^j_i \right)_{i \in I}$$

$$= \left( (f \otimes 1_{M_i}) \left( \sum_{\text{finite}} n_j \otimes m^j_i \right) \right)_{i \in I}$$
Then we know that 
\[(f \otimes 1_{M_i}) \left( \sum_{\text{finite}} n_j \otimes m_i^j \right) = 0, \quad \forall i \in I.\]

Since \( f \otimes 1_{M_i} \) is injective, then 
\[\sum_{\text{finite}} n_j \otimes m_i^j = 0, \quad \forall i \in I.\]

Which implies that 
\[\sum_{\text{finite}} n_j \otimes (m_i^j)_{i \in I} = 0.\]

Hence \( f \otimes 1_M \) is injective. Therefore, \( M \) is flat.

\[\square\]

5. Let \( A[x] \) be the ring of polynomials in one indeterminate over a ring \( A \). Prove that \( A[x] \) is a flat \( A \)-algebra.

\textbf{Proof.} We know that \( A[x] \) is a ring such that \( A \) is a subring of \( A[x] \), which implies that \( A[x] \) is an \( A \)-module. So for all \( i \geq 0 \), \( Ax^i \) is an \( A \)-module generated by \( x^i \) in \( A[x] \).

\textbf{Claim I:} \( Ax^i \cong A \) as \( A \)-modules.

Define \( \phi : A \to Ax^i \) as \( \phi(a) = ax^i \), it is easy to see that \( \phi \) is a bijective \( A \)-module homomorphism (Since \( ax^i = 0 \) iff \( a = 0 \)), so \( Ax^i \cong A \) as \( A \)-modules. Since \( A \) is a flat \( A \)-module, then \( Ax^i \) is also flat as \( A \)-module for all \( i \geq 0 \). On the other hand, since \( A[x] = \bigoplus_{i=0}^{\infty} Ax^i \), as \( A \)-modules.

By the result of the Problem 4, we know that \( A[x] \) is a flat \( A \)-module. Let \( i : A \to A[x] \) be the embedding of rings, that is, \( i(a) = a \) for all \( a \in A \), then \( A[x] \) is an \( A \)-algebra. Hence we know that \( A[x] \) is a flat \( A \)-algebra.

\[\square\]

6. For any \( A \)-module \( M \), let \( M[x] \) denote the set of all polynomials in \( x \) with coefficients in \( M \), that is to say expressions of the form
\[m_0 + m_1x + \cdots + m_rx^r, \quad m_i \in M.\]

Defining the product of an element of \( A[x] \) and an element of \( M[x] \) in the obvious way, show that \( M[x] \) is an \( A[x] \)-module. Show that \( M[x] \cong A[x] \otimes_A M \).

\textbf{Proof.} For any \( \sum_{i=0}^{t} a_i x^i \in A[x] \) and \( \sum_{i=0}^{r} m_i x^i \in M[x] \), let
\[\left( \sum_{i=0}^{t} a_i x^i \right) \cdot \left( \sum_{i=0}^{r} m_i x^i \right) = \sum_{j=0}^{t+r} \left( \sum_{i+j=i} a_j m_{i} \right) x^i.\]

\textbf{Claim I:} \( M[x] \) is an \( A[x] \)-module

It is easy to see that \( M[x] \) is an additive group, and the above scalar multiplication by \( A[x] \) is well defined. For all \( \sum_{i=0}^{r} m_i x^i \in M[x] \), we have
\[1 \cdot \left( \sum_{i=0}^{r} m_i x^i \right) = \sum_{i=0}^{r} m_i x^i.\]
It is easy to see that the distribution laws hold for this scalar multiplication. Now we only need to check the associativity law. In fact, for any \( \sum_{i=0}^{t} a_i x^i, \sum_{i=0}^{s} b_i x^i \in A[x] \) and any \( \sum_{i=0}^{r} m_i x^i \in M[x] \), we know that
\[
\left[ \left( \sum_{i=0}^{t} a_i x^i \right) \left( \sum_{i=0}^{s} b_i x^i \right) \right] \cdot \left( \sum_{i=0}^{r} m_i x^i \right) = \left[ \sum_{i=0}^{t+s} \left( \sum_{j_1+j_2=i} a_{j_1} b_{j_2} \right) x^i \right] \cdot \left( \sum_{i=0}^{r} m_i x^i \right)
\]
\[
= \sum_{i=0}^{t+s+r} \left( \sum_{j_4+j_5=i} \left( \sum_{j_1+j_2=j_3} a_{j_1} b_{j_2} m_{j_3} \right) \right) x^i
\]
\[
= \sum_{i=0}^{t+s+r} \left( \sum_{j_1+j_2+j_3=i} a_{j_1} b_{j_2} m_{j_3} \right) x^i
\]
\[
= \left[ \left( \sum_{i=0}^{t} a_i x^i \right) \left( \sum_{i=0}^{s} b_i x^i \right) \right] \cdot \left( \sum_{i=0}^{r} m_i x^i \right)
\]

In summary, we know that \( M[x] \) is an \( A[x] \)-module.

**Claim II:** \( M[x] \cong A[x] \otimes_A M \) as \( A[x] \)-modules.

Define the map \( \phi : A[x] \times M \to M[x] \) as: for all \( \sum_{i=0}^{r} a_i x^i \in A[x] \) and all \( m \in M \), we have
\[
\phi \left( \sum_{i=0}^{r} a_i x^i, m \right) = \sum_{i=0}^{r} (a_i m) x^i
\]
It is easy to see that \( \phi \) is well defined an \( A \)-bilinear map, by the universal property of tensor product, then there exists a unique \( A \)-module homomorphism \( \Phi : A[x] \otimes_A M \to M[x] \) such that for all \( \sum_{i=0}^{t} a_i x^i \in A[x] \) and all \( m \in M \), we have
\[
\Phi \left( \left( \sum_{i=0}^{t} a_i x^i \right) \otimes m \right) = \sum_{i=0}^{t} (a_i m) x^i
\]
Now we need to check that \( \Phi \) is an \( A[x] \)-module homomorphism, it suffices to check the \( A[x] \)-linearity for the simple tensors. In fact, for all \( \sum_{i=0}^{t} a_i x^i, \sum_{i=0}^{s} b_i x^i \in A[x] \) and \( m \in M \), we have
\[
\Phi \left( \left( \sum_{i=0}^{s} b_i x^i \right) \left( \left( \sum_{i=0}^{t} a_i x^i \right) \otimes m \right) \right) = \Phi \left( \left( \sum_{i=0}^{s} b_i x^i \right) \left( \sum_{i=0}^{t} a_i x^i \right) \right) \otimes m
\]
\[ \Phi \left( \sum_{i=0}^{t+s} \left( \sum_{j_1+j_2=i} b_{j_1} a_{j_2} \right) x^i \otimes m \right) \]

\[ = \sum_{i=0}^{t+s} \left( \sum_{j_1+j_2=i} b_{j_1} a_{j_2} \right) mx^i \]

\[ = \left( \sum_{i=0}^{t+s} \left( \sum_{j_1+j_2=i} b_{j_1} a_{j_2} \right) x^i \right) \cdot m \]

\[ = \left[ \left( \sum_{i=0}^{s} b_i x^i \right) \left( \sum_{i=0}^{t} a_i x^i \right) \right] \cdot m \]

\[ = \left( \sum_{i=0}^{s} b_i x^i \right) \cdot \left[ \left( \sum_{i=0}^{t} a_i x^i \right) \cdot m \right] \]

\[ = \left( \sum_{i=0}^{s} b_i x^i \right) \cdot \Phi \left( \left( \sum_{i=0}^{t} a_i x^i \otimes m \right) \right) \]

Also the additivity follows from \( A \)-module homomorphism. Hence \( \Phi \) is an \( A[x] \)-module homomorphism. Define \( \Psi : M[x] \rightarrow A[x] \otimes_A M \) as: for all \( \sum_{i=0}^{r} m_i x^i \in M[x] \), we have

\[ \Psi \left( \sum_{i=0}^{r} m_i x^i \right) = \sum_{i=0}^{r} x^i \otimes m_i \]

It is easy to see that \( \Psi \) is a well defined additive group homomorphism, now we need to check \( A[x] \)-linearity. For any \( \sum_{i=0}^{t} a_i x^i \in A[x] \) and \( \sum_{i=0}^{r} m_i x^i \in M[x] \), then

\[ \Psi \left( \sum_{i=0}^{t} a_i x^i \right) \cdot \left( \sum_{i=0}^{r} m_i x^i \right) = \Psi \left( \sum_{i=0}^{t+r} \left( \sum_{j_1+j_2=i} a_{j_1} m_{j_2} \right) x^i \right) \]

\[ = \sum_{i=0}^{t+r} x^i \otimes \left( \sum_{j_1+j_2=i} a_{j_1} m_{j_2} \right) \]

\[ = \sum_{i=0}^{t+r} \sum_{j_1+j_2=i} x^i \otimes (a_{j_1} m_{j_2}) \]

\[ = \sum_{i=0}^{t+r} \sum_{j_1+j_2=i} (a_{j_1} x^i) \otimes m_{j_2} \]

\[ = \sum_{i=0}^{t+r} \sum_{j_1+j_2=i} ((a_{j_1} x^{j_1}) x^{j_2}) \otimes m_{j_2} \]

\[ = \sum_{i=0}^{t+r} \sum_{j_1+j_2=i} (a_{j_1} x^{j_1}) \cdot (x^{j_2} \otimes m_{j_2}) \]

\[ = \sum_{i=0}^{t+r} \sum_{j_1+j_2=i} (a_{j_1} x^{j_1}) \cdot \Psi((m_{j_2} x^{j_2}) \]
\[= \left( \sum_{i=0}^{t} \alpha_i x_i^t \right) \cdot \Psi \left( \sum_{i=0}^{r} m_i x^i \right) \]

Hence \( \Psi \) is \( A[x] \)-module homomorphism. Now for any \( \sum_{i=0}^{r} m_i x^i \in M[x] \), then

\[
\Phi \circ \Psi \left( \sum_{i=0}^{r} m_i x^i \right) = \Phi \left( \sum_{i=0}^{r} x^i \otimes m_i \right) = \sum_{i=0}^{r} \Phi(x^i \otimes m_i) = \sum_{i=0}^{r} m_i x^i.
\]

That is, \( \Phi \circ \Psi = \text{Id} \). Now for any \( \sum_{i=0}^{r} \alpha_i x^i \in A[x] \) and all \( m \in M \), we have

\[
\Psi \circ \Phi \left( \left( \sum_{i=0}^{r} \alpha_i x^i \right) \otimes m \right) = \Psi \left( \sum_{i=0}^{r} (\alpha_i m) x^i \right) = \sum_{i=0}^{r} x^i \otimes (\alpha_i m) = \sum_{i=0}^{r} (\alpha_i x^i) \otimes m = \left( \sum_{i=0}^{r} (\alpha_i x^i) \right) \otimes m.
\]

Which implies that \( \Psi \circ \Phi = \text{Id} \). Therefore, we know that \( \Phi \) and \( \Psi \) are \( A[x] \)-module isomorphisms, in particular, \( M[x] \cong A[x] \otimes_A M \) as \( A[x] \)-modules.

\[\Box\]

15. Let \( A \) be a ring and let \( X \) be the set of all prime ideals of \( A \). For each subset \( E \) of \( A \), let \( V(E) \) denote the set of all prime ideals of \( A \) which contain \( E \). Prove that

a. If \( a \) is the ideal generated by \( E \), then \( V(E) = V(a) = V(\sqrt{a}) \).

b. \( V(0) = X \), \( V(1) = \emptyset \).

c. If \( (E_i)_{i \in I} \) is any family of subsets of \( A \), then

\[V \left( \bigcup_{i \in I} E_i \right) = \bigcap_{i \in I} V(E_i).\]

d. \( V(a \cap b) = V(ab) = V(a) \cup V(b) \) for any ideals \( a, b \) of \( A \).

\textit{Proof.} a. Since \( E \subset a \subset \sqrt{a} \), then

\[V(\sqrt{a}) \subset V(a) \subset V(E).\]

Now for any prime ideal \( p \) of \( A \) such that \( E \subset p \), by the definition of \( a \), then \( a \subset p \), that is, \( p \in V(a) \). Also since \( a \subset p \), then \( \sqrt{a} \subset \sqrt{p} \). Since \( p \) is prime, then \( \sqrt{p} = p \). Hence \( \sqrt{a} \subset p \), that is, \( p \in V(\sqrt{p}) \).

Therefore, we know that

\[V(\sqrt{a}) = V(a) = V(E).\]
b. For any prime ideal \( p \) of \( A \), we know that \( 0 \in p \), then \( p \in V(0) \). Hence
\[
V(0) = X.
\]
For \( V(1) \), we must have \( V(1) = \emptyset \), otherwise, there exists some prime ideal \( p \) of \( A \) such that \( 1 \in p \), which implies that \( p = A \), contradiction. Hence
\[
V(1) = \emptyset.
\]
c. Since for \( i \in I \), we have \( E_i \subset \bigcup_{i \in I} E_i \), then
\[
V \left( \bigcup_{i \in I} E_i \right) \subset V(E_i), \quad \forall i \in I.
\]
Hence
\[
V \left( \bigcup_{i \in I} E_i \right) \subset \bigcap_{i \in I} V(E_i).
\]
On the other hand, for all \( p \in \bigcap_{i \in I} V(E_i) \), then
\[
p \in V(E_i), \quad \forall i \in I.
\]
That is,
\[
E_i \subset p, \quad \forall i \in I.
\]
Hence
\[
\bigcup_{i \in I} E_i \subset p.
\]
That is, \( p \in V \left( \bigcup_{i \in I} E_i \right) \). Therefore, we know that
\[
V \left( \bigcup_{i \in I} E_i \right) = \bigcap_{i \in I} V(E_i).
\]
d. For any ideals \( a, b \) of \( A \), then
\[
ab \subset a \bigcap b \subset a, \quad \text{and} \quad ab \subset a \bigcap b \subset b.
\]
So we have
\[
V(a) \subset V(a \bigcap b) \subset V(ab), \quad \text{and} \quad V(a) \subset V(a \bigcap b) \subset V(ab).
\]
Hence
\[
V(a) \bigcup V(b) \subset V(a \bigcap b) \subset V(ab).
\]
Now for any \( p \in V(ab) \), then \( ab \subset p \) and \( p \) is prime ideal, which implies that \( a \subset p \) or \( b \subset p \), that is, \( p \in V(a) \bigcup V(b) \). Therefore, we get
\[
V(a) \bigcup V(b) = V(a \bigcap b) = V(ab)
\]
16. Draw pictures of \( \text{Spec}(\mathbb{Z}) \), \( \text{Spec}(\mathbb{R}) \), \( \text{Spec}(\mathbb{C}[x]) \), \( \text{Spec}(\mathbb{R}[x]) \) and \( \text{Spec}(\mathbb{Z}[x]) \).
Proof. a. For $\mathbb{Z}$ which is a PID, the ideal $p$ of $\mathbb{Z}$ is prime if and only if $p = 0$ or $p = p\mathbb{Z}$ for some prime number $p$ in $\mathbb{Z}$, that is,

$$\text{Spec } \mathbb{Z} = \{ p\mathbb{Z} : p \text{ is a prime number in } \mathbb{Z} \text{ or } p = 0 \}$$

Since $\mathbb{Z}$ is a PID, then for any ideal $a \in \mathbb{Z}$ with $a \neq 0$ and $a \neq \mathbb{Z}$, there exists a unique $m \geq 2 \in \mathbb{N}$ such that $a = m\mathbb{Z}$. For $m$, by the fundamental theorem of arithmetic, there exists a unique prime factorization

$$m = p_1^{e_1} \cdots p_k^{e_k}, \quad e_i \geq 1.$$ 

Then we know that for all $1 \leq i \leq k$, the ideal $p_i = p_i\mathbb{Z} \subseteq \text{Spec}\mathbb{Z}$ and

$$V(a) = V(m\mathbb{Z}) = \{ p_1, \ldots, p_k \}.$$ 

That is to say that nontrivial closed sets in Spec $\mathbb{Z}$ is a finite collection of prime ideals in $\mathbb{Z}$. On the other hand, for any finite collection of prime ideals $\{ p_1, \ldots, p_k \}$ in $\mathbb{Z}$, for each $1 \leq i \leq k$, there exists a unique prime number $p_i \in \mathbb{Z}$ such that $p_i = p_i\mathbb{Z}$. Let $m = p_1 \cdots p_k$, and $a = m\mathbb{Z}$, then

$$V(m\mathbb{Z}) = V(a) = \{ p_1, \ldots, p_k \}.$$ 

So we know that a subset $U$ of Spec $\mathbb{Z}$ is open if and only if $U = \emptyset$ or Spec $\mathbb{Z}\setminus U$ is a finite set. That is to say, the topology on Spec $\mathbb{Z}$ is the finite completion topology.

b. For $\mathbb{R}$, since $\mathbb{R}$ is a field, then only prime ideal in $\mathbb{R}$ is 0, that is,

$$\text{Spec } \mathbb{R} = \{ \emptyset \}.$$ 

The open sets of Spec $\mathbb{R}$ are $\emptyset$ and $\{ \} = \{ a \}$, and the topology on Spec $\mathbb{R}$ is the discrete topology.

c. For $\mathbb{C}[x]$, since $\mathbb{C}[x]$ is PID, then the ideal $p$ of $\mathbb{C}[x]$ is prime if and only if $p = 0$ or $p = f(x)\mathbb{C}[x]$ for some monic irreducible polynomial $f(x) \in \mathbb{C}[x]$ with $\text{deg}f(x) \geq 1$. Since $\mathbb{C}$ is algebraic closed, then only monic irreducible polynomials are of the form $x - c$ for some $c \in \mathbb{C}$. Hence we know that

$$\text{Spec } \mathbb{C}[x] = \{ p : p = 0 \text{ or } p = (x - c)\mathbb{C}[x] \text{ for some } c \in \mathbb{C} \}.$$ 

Since $\mathbb{C}[x]$ is a PID, then for any ideal $a \in \mathbb{C}[x]$ with $a \neq 0$ and $a \neq \mathbb{C}[x]$, there exists a unique monic polynomial $m(x) \in \mathbb{C}[x]$ such that $a = m(x)\mathbb{C}[x]$. For $m(x)$, since $\mathbb{C}[x]$ is UFD, then there exists some $c_1, c_2, \cdots, c_k \in \mathbb{C}$ such that

$$m(x) = (x - c_1)^{e_1} \cdots (x - c_k)^{e_k}, \quad e_i \geq 1.$$ 

Then we know that for all $1 \leq i \leq k$, the ideal $p_i = (x - c_1)\mathbb{C}[x] \subseteq \text{Spec}\mathbb{C}[x]$ and

$$V(a) = V(m(x)\mathbb{C}[x]) = \{ p_1, \ldots, p_k \}.$$ 

That is to say that nontrivial closed sets in Spec $\mathbb{C}[x]$ is a finite collection of prime ideals in $\mathbb{C}[x]$. On the other hand, for any finite collection of prime ideals $\{ p_1, \ldots, p_k \}$ in $\mathbb{C}[x]$, for each $1 \leq i \leq k$, there exists a unique $c_i \in \mathbb{C}$ such that $p_i = (x - c_i)\mathbb{C}[x]$. Let $m(x) = (x - c_1) \cdots (x - c_k)$, and $a = m(x)\mathbb{C}[x]$, then

$$V(m(x)\mathbb{C}[x]) = V(a) = \{ p_1, \ldots, p_k \}.$$ 

So we know that a subset $U$ of Spec $\mathbb{C}[x]$ is open if and only if $U = \emptyset$ or Spec $\mathbb{C}[x]\setminus U$ is a finite set. That is to say, the topology on Spec $\mathbb{C}[x]$ is the finite completion topology.

d. For $\mathbb{R}[x]$, since $\mathbb{R}[x]$ is PID, then the ideal $p$ of $\mathbb{R}[x]$ is prime if and only if $p = 0$ or $p = f(x)\mathbb{R}[x]$ for some monic irreducible polynomial $f(x) \in \mathbb{R}[x]$ with $\text{deg}f(x) \geq 1$. Since only monic irreducible polynomials are of the form $x - c$ for some $c \in \mathbb{R}$ or $x^2 + ax + b$ with $a^2 - 4b < 0$ for some $a, b \in \mathbb{R}$. Hence we know that

$$\text{Spec } \mathbb{R}[x] = \{ p : p = 0 \text{ or } p = (x - c)\mathbb{R}[x] \text{ for } c \in \mathbb{R} \text{ or } p = (x^2 + ax + b)\mathbb{R}[x] \text{ for } a, b \in \mathbb{R} \text{ with } a^2 - 4b < 0 \}.$$
Since $R[x]$ is a PID, then for any ideal $a \in R[x]$ with $a \neq 0$ and $a \neq R[x]$, there exists a unique monic polynomial $m(x) \in R[x]$ such that $a = m(x)R[x]$. For $m(x)$, since $R[x]$ is UFD, then there exists some irreducible monic polynomials $p_1(x), \ldots, p_k(x) \in R[x]$ such that

$$m(x) = p_1(x)^{e_1} \cdots p_k(x)^{e_k}, \quad e_i \geq 1.$$ 

Then we know that for all $1 \leq i \leq k$, the ideal $p_i = p_i(x)R[x] \in \text{Spec}R[x]$ and

$$V(a) = V(m(x)R[x]) = \{p_1, \ldots, p_k\}.$$ 

That is to say that nontrivial closed sets in $\text{Spec} R[x]$ is a finite collection of prime ideals in $R[x]$. On the other hand, for any finite collection of prime ideals $\{p_1, \ldots, p_k\} \in R[x]$, for each $1 \leq i \leq k$, there exists a unique monic irreducible polynomial $p_i(x) \in R[x]$ such that $p_i = p_i(x)R[x]$. Let $m(x) = p_1(x) \cdots p_k(x)$, and $a = m(x)R[x]$, then

$$V(m(x)R[x]) = V(a) = \{p_1, \ldots, p_k\}.$$ 

So we know that a subset $U$ of $\text{Spec} R[x]$ is open if and only if $U = \emptyset$ or $\text{Spec} R[x] \setminus U$ is a finite set. That is to say, the topology on $\text{Spec} R[x]$ is the finite completion topology.

e. Claim I: The ideal $p$ of $Z[x]$ is prime if and only if $p$ is one of the following cases:

i. $p = 0$.

ii. $p = (p)$ for some prime number $p$ in $\mathbb{Z}$.

iii. $p = (f(x))$ for some primitive irreducible polynomial $f(x)$ in $\mathbb{Z}[x]$.

iv. $p = (p, f(x))$ for some prime number $p$ in $\mathbb{Z}$ and primitive irreducible polynomial $f(x)$ in $\mathbb{Z}[x]$ such that $f(x)$ is also irreducible in $\mathbb{Z}[x]/p\mathbb{Z}[x] \cong F_p[x]$.

($\Longleftarrow$) i. Since $Z[x]$ is a domain, then $p = 0$ is prime in $[Z[x]]$.

ii. For any $f(x), g(x) \in Z[x]$ such that $f(x)g(x) \in p = (p)$ for some prime number $p$ in $\mathbb{Z}$, then

$$p | f(x)g(x)$$

Recall the Gauss’s Lemma:

Let $A$ be a UFD, $f(x)$ and $g(x)$ be primitive polynomials in $A[X]$, then $f(x)g(x)$ is also primitive.

Since $p$ is a prime number in $\mathbb{Z}$, by the Gauss’s Lemma, we know that $p | f(x)$ or $p | g(x)$ in $\mathbb{Z}[x]$, that is, $f(x) \in p$ or $g(x) \in p$. Hence $p$ is prime in $\mathbb{Z}[x]$.

iii. For any $g(x), h(x) \in \mathbb{Z}[x]$ such that $g(x)h(x) \in p = (f(x))$ for some primitive irreducible polynomial $f(x)$ in $\mathbb{Z}[x]$, then

$$f(x)|g(x)h(x)$$

Since $f(x)$ is irreducible in $\mathbb{Z}[x]$, then $f(x)$ is also irreducible in $Q[x]$. Hence $f(x)|g(x)$ or $f(x)|h(x)$ in $Q[x]$. Without loss of generality, assume $f(x)|g(x)$ in $Q[x]$, then there exists some $m(x) \in Q[x]$ such that

$$g(x) = m(x)f(x).$$

Since $f(x), g(x) \in \mathbb{Z}[x]$ and $f$ is primitive, by the Gauss’s Lemma, then $m(x) \in \mathbb{Z}[x]$, that is, $f(x)|g(x)$ in $\mathbb{Z}[x]$. Hence $p$ is prime in $\mathbb{Z}[x]$.

iv. $p = (p, f(x))$ for some prime number $p$ in $\mathbb{Z}$ and primitive irreducible polynomial $f(x)$ in $\mathbb{Z}[x]$ such that $f(x)$ is also irreducible in $\mathbb{Z}/p\mathbb{Z}[x]$. Let $\pi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the natural ring homomorphism, that is, for all $n \in \mathbb{Z}$, we have

$$\pi(n) = n + p\mathbb{Z}.$$ 

Then $\pi$ can induce a ring homomorphism $\overline{\pi} : \mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z}[x]$ such that $\overline{\pi}|_\mathbb{Z} = \pi$. Since $\mathbb{Z}/p\mathbb{Z}$ is a field and $\overline{f}(x)$ is irreducible on $\mathbb{Z}/p\mathbb{Z}[x]$, then $\mathbb{Z}/p\mathbb{Z}[x]/(\overline{f}(x))$ is a field extension of $\mathbb{Z}/p\mathbb{Z}$. 

Define the map $\Phi : \mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z}[x]/(\overline{f(x)})$ as: for all $g(x) \in \mathbb{Z}[x]$, we have

$$\Phi(g(x)) = \overline{g(x)} + (\overline{f(x)}).$$

It is easy to see that $\Phi$ is a ring homomorphism. Now let’s look at the kernel of $\Phi$. It is easy to see that $p, f(x) \in \ker \Phi$, since $\ker \Phi$ is an ideal of $\mathbb{Z}[x]$, then

$$(p, f(x)) \subset \ker \Phi.$$

On the other hand, for all $g(x) \in \ker \Phi$, then $\overline{g(x)} \in \overline{f(x))}$. That is, there exists some $h \in \mathbb{Z}[x]$ such that

$$\overline{g(x)} = \overline{f(x)}h(x) = \overline{f(x)h(x)}.$$

Hence $\overline{g(x)} - \overline{f(x)h(x)} = 0$, that is, there exists some $k(x) \in \mathbb{Z}[x]$ such that

$$g(x) - f(x)h(x) = pk(x).$$

That is, $g(x) = h(x)f(x) + k(x)p \in (p, f(x))$. Hence we get

$$\ker \Phi = (p, f(x)).$$

By the first isomorphism theorem, then

$$\mathbb{Z}/p\mathbb{Z}[x]/(\overline{f(x)}) \cong \mathbb{Z}[x]/(p, f(x)),$$

which is a field. Hence $(p, f(x))$ is maximal in $\mathbb{Z}[x]$, in particular, $p = (p, f(x))$ is prime in $\mathbb{Z}[x]$.

Now assume $p$ is a prime ideal in $\mathbb{Z}[x]$. If $p = 0$, we are done. Now assume $p \neq 0$. Let $q = p \cap \mathbb{Z}$, then $q$ is prime in $\mathbb{Z}$.

Case I: If $q = 0$. Let $S = \mathbb{Z}\setminus \{0\}$, then $S$ is a multiplicative subset of $\mathbb{Z}[x]$ and $p \cap S = \emptyset$. Since $S^{-1}\mathbb{Z} = \mathbb{Q}$, then

$$S^{-1}\mathbb{Z}[x] = \mathbb{Q}[x].$$

Since $S \cap p = \emptyset$ and $p$ is prime in $\mathbb{Z}[x]$, then $S^{-1}p$ is prime in $S^{-1}\mathbb{Z}[x] = \mathbb{Q}[x]$. Since $\mathbb{Q}[x]$ is PID, then there exists some irreducible polynomial $f(x) \in \mathbb{Q}[x]$ such that $S^{-1}p = (f(x))$ in $\mathbb{Q}[x]$. Then by multiplying some constant, without loss of generality, we can assume $f(x) \in \mathbb{Z}[x]$. Since $p \cap S = \emptyset$, then

$$p = (f(x)) \cap \mathbb{Z}[x].$$

That is, $p = (f(x))$ in $\mathbb{Z}[x]$, where $f(x)$ is primitive irreducible polynomial in $\mathbb{Z}[x]$.

Case II: If $q \neq 0$. Since $q$ is prime in $\mathbb{Z}$, then there exists some prime number $p$ in $\mathbb{Z}$ such that $q = p\mathbb{Z}$, then $p\mathbb{Z}[x] \subset p$. By the forth isomorphism theorem, we know that $p/p\mathbb{Z}[x]$ is a prime ideal in $\mathbb{Z}[x]/p\mathbb{Z}[x] = \mathbb{Z}/p\mathbb{Z}[x]$. Since $\mathbb{Z}/p\mathbb{Z}$ is a field, then $\mathbb{Z}[x]/p\mathbb{Z}[x] = \mathbb{Z}/p\mathbb{Z}[x]$ is PID.

Subcase I: $p/p\mathbb{Z}[x] = 0$, then $p = (p)$, we are done.

Subcase II: $p/p\mathbb{Z}[x] \neq 0$, since $p/p\mathbb{Z}[x]$ is a prime ideal in $\mathbb{Z}[x]/p\mathbb{Z}[x] = \mathbb{Z}/p\mathbb{Z}[x]$ which is PID, then there exists some primitive irreducible polynomial $f(x) \in \mathbb{Z}[x]$ such that $\overline{f(x)}$ is irreducible in $\mathbb{Z}/p\mathbb{Z}[x]$ and

$$p/p\mathbb{Z}[x] = (\overline{f(x)}).$$

Hence we get $p = (p, f(x))$.

In summary, we can conclude that the Claim I is true.

\[\square\]