THE EQUIVALENCE OF TWO WAYS OF COMPUTING DISTANCES FROM DISSIMILARITIES FOR ARBITRARY SETS OF STIMULI

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Abstract. Given a set endowed with pairwise dissimilarities, the Dissimilarity Cumulation procedure computes the (quasi)distance between any two elements of the set as the infimum of the sums of dissimilarities across all finite chains of elements connecting the two elements. For finite sets this procedure is known to be equivalent to recursive corrections for violations of the triangle inequality in any sequence of ordered triads of points which contains every triad a sufficient number of times. This paper extends this equivalence to infinite sets.

For a finite stimulus set $\mathcal{S}$, a dissimilarity is a function $D: \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ which is nonnegative and equal to zero if and only if its two arguments coincide. The dissimilarity $D(a, b)$, which we will write as $D_{ab}$, is usually an empirically observable quantity, such as the mean numerical estimate of dissimilarity by an observer, or, in Fechnerian scaling, either of the two “psychometric increments”

$$\Pr[a \text{ is judged different from } b] - \Pr[a \text{ is judged different from } a]$$

and

$$\Pr[b \text{ is judged different from } a] - \Pr[a \text{ is judged different from } a],$$

where the probabilities are defined on $\mathcal{S} \times \mathcal{S}$ following certain “canonical” transformation of the stimuli (Dzhafarov, 2002; Dzhafarov and Colonius, 2006). While by itself $D$ imposes a rather weak structure on the set $\mathcal{S}$, it allows one to compute, for each pair of stimuli $a, b$, a quantity interpretable as a “subjective distance from $a$ to $b$.” This is done as follows. Denoting by $X = x_1 \cdots x_n$ ($n \geq 1$) a finite sequence of stimuli (referred to as a chain and written as a string, without commas), denoting by $aXb$ the chain $ax_1 \cdots x_nb$, and putting

$$D_{aXb} = D_{ax_1} + \sum_{1 \leq i \leq n-2} D_{x_ix_{i+1}} + D_{x_{n-1}b},$$

we define

$$G_{ab} = \min_X D_{aXb},$$

where we write $G_{ab}$ in place of $G(a, b)$, and the minimum is taken over all chains $X$ in $\mathcal{S}$. The function $G$ is easily seen to be a (quasi)metric, that is, a dissimilarity function that satisfies the triangle inequality (but which is not necessarily

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symmetric). It is referred to as the \textit{(quasi)}metric induced by the dissimilarity \(D\).\footnote{The reason for writing the qualifier “quasi” in parentheses is that \(G\) is a quasimetric in traditional terminology but also, as explained below, a metric in the nomenclature of Dissimilarity Cumulation theory (see Dzhafarov, 2010b).} A symmetrization, if needed, can be obtained by taking \(Dab + Dba\), and will not concern us in this paper.

It is shown in Dzhafarov (2010a) that \(Gab\) can be also be computed from \(Dab\) by means of another procedure. We present it in a modified form to better link it to the construction we introduce in the next section. Let \(\text{Tri} (S)\) be the set of all ordered triples \((a, b, c)\in S^3\) with \(a \neq b, a \neq c,\) and \(b \neq c\). Call each such triple a triad. Consider any sequence \(T\) of triads in which every triad occurs an infinite number of times. Suppose that we move along \(T\) from one triad to another and every time when we find that \(Dab > Dac + Dcb\) (i.e., the triangle inequality is violated), we replace \(Dab\) with \(Dac + Dcb\) and consider this sum a new, redefined value of \(Dab\). Then, after a finite number of such steps the redefined \(D\) will coincide with the quasimetric \(G\) induced by the original \(D\) (whence subsequent steps will no longer change \(D\)). The number of these steps can be arbitrarily large, but, as it is known from the Floyd-Warshall algorithm (Floyd, 1962), it can be made no larger than \(n^3\), where \(n\) is the cardinality of \(S\).\footnote{In the Floyd-Warshall algorithm, having enumerated the elements of \(S\) 1 to \(n\), the replacement of \(Dab\) with \(\min \{Dab, Dac + Dcb\}\) occurs within three nested cycles (each combination whereof we call a step): \(c = 1\) to \(n\), nesting \(a = 1\) to \(n\), nesting \(b = 1\) to \(n\). At the end of this triple-cycle all violations of the triangle inequality are guaranteed to be corrected. If one excludes degenerate triads, the number of steps in the triple-cycle is \(n(n-1)(n-2)\).}

It is not immediately obvious how this recursive procedure of correcting for violations of the triangle inequality should be defined in the case of an infinite stimulus set \(S\); and if appropriately defined, whether for infinite sets too the “eventual” result of such corrections is guaranteed to be achieved and coincide with the metric induced by the original dissimilarity in accordance with Dissimilarity Cumulation (DC) theory (Dzhafarov & Colonius, 2007; Dzhafarov, 2008a-b, 2009, 2010b). We show in this paper that the answer to the latter question is affirmative after the correction procedure has been extended to arbitrary sets by means of transfinite recursion. We provide a general account of ordinals and transfinite recursion, sufficient for our purposes, in Appendix A.

1. Dissimilarity and Related Notions for Arbitrary Sets

For an arbitrary stimulus set \(S\), we follow notation conventions already used in the introduction. Chains \(X\) are finite sequences of elements of \(S\), written as strings: \(ab, abc, x_1 \cdots x_n,\) etc. \(XY\) is the concatenation of \(X\) and \(Y\), so we can write \(aXb, aXbYZc,\) etc. For any function \(F : S \times S \to \mathbb{R}\) the notation \(FX\) denotes 0 if \(X\) is the empty chain or a chain of length 1, and denotes \(Fx_1x_2 + \cdots + Fx_{n-1}x_n\) if \(X = x_1 \cdots x_n, n \geq 2\). Let \(S\) denote the set of all chains in \(S\), including the empty chain.

We call any function \(D : S \times S \to \mathbb{R}\) which is nonnegative and equal to zero if and only if its two arguments coincide a \textit{pre-dissimilarity function}. The minimum in (2) need not generally exist and has to be replaced with

\begin{equation}
Gab = \inf_{X \in S} DaXb.
\end{equation}
The function $G_{ab}$ is nonnegative, equal to zero at $a = b$, and easily seen to satisfy the triangle inequality: by definition, for any $X, Y \in \mathcal{S}$,

$$G_{ab} \leq D_{aXcYb} = D_{aXc} + D_{cYb},$$

so $G_{ab} \leq G_{ac} + G_{cb}$. The function $G$ is not itself, however, a pre-dissimilarity function, as $G_{ab} = 0$ does not imply $a = b$. Consider, for instance, $D_{ab} = (b - a)^2$ with $\mathcal{G} = \mathbb{R}$: in this case

$$G_{ab} = \inf_{x_1, x_2, \ldots, x_n} (a - x_0)^2 + \sum_{1 \leq i \leq n-1} (x_{i+1} - x_i)^2 + (b - x_n)^2$$

$$\leq \inf_{i \geq 1} \sum_{1 \leq i \leq k} ((a + (i - 1)\frac{b-a}{k}) - (a + i\frac{b-a}{k}))^2 = \inf_{i \geq 1} \sum_{1 \leq i \leq k} \frac{(b-a)^2}{k^2} = 0,$$

for any reals $a$ and $b$. The function $G$ defined by (3) therefore is a pseudo-quasi-metric ($p.q.$-metric, for short) induced by the pre-dissimilarity $D$. For any $a, b \in \mathcal{G}$, the value $G_{ab}$ will be referred to as the $p.q.$-distance from $a$ to $b$.

A metric, in the nomenclature of DC theory, is a $p.q.$-metric $M$ such that $M_{ab} > 0$ for any $a \neq b$, and $M_{an}b_n \to 0$ implies $M_{bn}a_n \to 0$ for any sequences $\{a_n\}$ and $\{b_n\}$ in $\mathcal{G}$. To ensure that the $p.q.$-metric $G$ induced by a pre-dissimilarity $D$ is a metric, the pre-dissimilarity should be strengthened by additional properties, making it a dissimilarity. In DC theory these properties are: (1) $D_{an}b_n - D_{a'n}b'_n \to 0$ for any sequences $\{a_n\}, \{a'_n\}, \{b_n\}, \{b'_n\}$ in $\mathcal{G}$ such that $D_{an}a'_n \to 0$ and $D_{bn}b'_n \to 0$ (uniform continuity), and (2) $D_{an}b_n \to 0$ for any sequences $\{a_n\}, \{b_n\}$ in $\mathcal{G}$ such that $D_{an}x_nb_n \to 0$ for some sequence $\{X_n\}$ in $\mathcal{G}$ (the “chain property”). For finite sets these properties are satisfied trivially, whence any pre-dissimilarity on a finite set is a dissimilarity, and the $p.q.$-metric induced by it is a metric.

We shall show below that a certain procedure of recursively redefining a pre-dissimilarity function $D$ will “eventually” (in a transfinite sense) always result in the $p.q.$-metric $G$ induced by $D$, whether this $p.q.$-metric is a metric or not. In other words, we will assume that an infinite stimulus set $\mathcal{G}$ is endowed with a pre-dissimilarity $D$ which induces the $p.q.$-metric $G$ according to (3). Then we will define a recursive procedure which consists in inspecting the ordered triads $abc$ of the elements of $\mathcal{G}$ one by one, and replacing $D_{ab}$ with $D_{ac} + D_{cb}$ if the former exceeds the latter, precisely as it was done for finite $\mathcal{G}$. The triads, however, will now have to be enumerated by transfinite numbers. We will prove that however this enumeration is performed, provided each triad is enumerated by a sufficiently large set of transfinite numbers, all violations of the triangle inequality in $\mathcal{G}$ will have been corrected at some transfinite step, at which step the corrected pre-dissimilarity $D$ will have been transformed into the $p.q.$-metric $G$. Moreover, we will estimate the cardinality of the set of the transfinite steps it will take to achieve $G$. We will show that if the inspections of the triads for violations of the triangle inequality are organized “economically,” the cardinality in question coincides with that of the stimulus set $\mathcal{G}$.

2. Recursive Corrections in Infinite Sets

We assume in this section that the stimulus set $\mathcal{G}$ is infinite. Denoting by $\text{Ord}$ the class of all ordinals, let $T : \text{Ord} \to \text{Tri}(\mathcal{G})$ be any class function such that for

\footnote{This “symmetry in the small” property replaces the global symmetry, $G_{ab} = G_{ba}$, as the latter plays no useful role in DC theory. This notion of a metric is common in Finsler geometry (from which DC theory has evolved), where the global symmetry is often dropped and the symmetry in the small is always satisfied.}
every $abc \in \text{Tri}(\mathcal{S})$ and every ordinal $\alpha$ there is an ordinal $\beta \geq \alpha$ with $T(\beta) = abc$. Fix a pre-dissimilarity $D : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$, and let $G$ be the induced p.q.-metric.

**Definition 1.** Define for each ordinal $\alpha$ a function $M^{(\alpha)} : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ by transfinite recursion:

1. $M^{(0)} = D$;
2. if $\alpha = \beta + 1$, then for all $a, b \in \mathcal{S}$,
   \[ M^{(\alpha)}_{ab} = \begin{cases} 
   \min\{M^{(\beta)}_{ab}, M^{(\beta)}_{ac} + M^{(\beta)}_{cb}\} & \text{if } T(\beta) = abc, \\
   M^{(\beta)}_{ab} & \text{otherwise}; 
   \end{cases} \]
3. if $\alpha$ is a limit ordinal, then $M^{(\alpha)} = \inf_{\beta < \alpha} M^{(\beta)}$.

It is obvious from this definition that $M^{(\alpha)}_{ab}$ is nonincreasing in $\alpha$ for any $a, b \in \mathcal{S}$.

**Lemma 2.** Let $\alpha$ be any ordinal.

1. For all $a, b \in \mathcal{S}$, $M^{(\alpha)}_{ab} \geq Gab$.
2. If $M^{(\alpha)}$ satisfies the triangle inequality, then for all $a, b \in \mathcal{S}$, $M^{(\alpha)}_{ab} = Gab$.

*Proof.* We prove part 1 by transfinite induction. It clearly holds for $\alpha = 0$, since $D \geq G$. Let it hold for all $\beta < \alpha$. It clearly follows from Definition 1(3) that it holds for $\alpha$ if it is a limit ordinal. If $\alpha = \beta + 1$ for some $\beta$, then, by part 2 of Definition 1, for all $a, b \in \mathcal{S}$, either
\[ M^{(\alpha)}_{ab} = M^{(\beta)}_{ab} \geq Gab, \]
or
\[ M^{(\alpha)}_{ab} = M^{(\beta)}_{ac} + M^{(\beta)}_{cb} \geq Gac + Gcb \geq Gab, \]
where the last inequality holds because $G$ satisfies the triangle inequality. This completes the proof.

To prove part 2, suppose $M^{(\alpha)}$ satisfies the triangle inequality. Then, for all $a, b \in \mathcal{S}$ and every $X \in \mathcal{S}$, we have
\[ M^{(\alpha)}_{ab} \leq M^{(\alpha)}_{aXb} \leq DaXb, \]
since $M^{(\alpha)} \leq M^{(0)} = D$. We conclude that
\[ M^{(\alpha)}_{ab} \leq \inf_{X \in \mathcal{S}} DaXb = Gab, \]
and the equality $M^{(\alpha)}_{ab} = Gab$ follows then from part 1. \qed

**Lemma 3.**

1. The class
   \[ C_{ab} = \{ \mu \in \text{Ord} : M^{(\mu+1)}_{ab} \neq M^{(\mu)}_{ab} \} \]
   is a countable set for every $a, b \in \mathcal{S}$.
2. The class
   \[ C = \{ \mu \in \text{Ord} : M^{(\mu)} \neq M^{(\mu+1)} \} \]
   is a set of cardinality $\leq |\mathcal{S}|$. 

Theorem 4. There exists an ordinal \( \alpha_M \) such that for all \( \alpha \geq \alpha_M \), and

(1) \( M^{(\alpha_M)} = M^{(\alpha)} \)

(2) \( M^{(\alpha_M)} \) satisfies the triangle inequality.

Thus, \( M^{(\alpha_M)} = G \).

Proof. Let the set \( C_{ab} \) be as in Lemma 3, and let \( \alpha_{ab} \) be the least ordinal larger than each \( \alpha \in C_{ab} \). For any \( \alpha \geq \alpha_{ab} \), we have \( M^{(\alpha+1)}_{ab} = M^{(\alpha)}_{ab} \), whence if \( M^{(\alpha)}_{ab} = M^{(\alpha_{ab})}_{ab} \) then \( M^{(\alpha+1)}_{ab} = M^{(\alpha_{ab})}_{ab} \). If \( \alpha \) is a limit ordinal and \( M^{(\beta)}_{ab} = M^{(\alpha_{ab})}_{ab} \) for all \( \alpha_{ab} \leq \beta < \alpha \), then

\[
M^{(\alpha)}_{ab} = \inf_{\beta < \alpha} M^{(\beta)}_{ab} = \inf_{\alpha_{ab} \leq \beta < \alpha} M^{(\beta)}_{ab} = M^{(\alpha_{ab})}_{ab},
\]

where the second equality holds because \( M^{(\gamma)}_{ab} \) is nonincreasing in \( \gamma \). By transfinite induction, it follows that for all \( \alpha \geq \alpha_{ab} \), \( M^{(\alpha)}_{ab} = M^{(\alpha_{ab})}_{ab} \). Now consider the class function \( f : \mathcal{S}^2 \rightarrow \text{Ord} \) given by \( f(a,b) = \alpha_{ab} \). As \( \mathcal{S}^2 \) is a set and \( \text{Ord} \) is a proper class, \( f \) must be bounded by some ordinal, and we let \( \alpha_M \) be the least such. It is readily seen that \( \alpha_M \) satisfies part 1 of the statement of the theorem.

To see that \( \alpha_M \) also satisfies part 2 of the statement, we argue by contradiction. Suppose there is a triad \( abc \) in \( \text{Tri}(\mathcal{S}) \) such that

\[
M^{(\alpha)}_{ab} > M^{(\alpha_M)}_{ac} + M^{(\alpha_M)}_{cb}.
\]

By definition of \( T \), there exists some \( \alpha \geq \alpha_M \) such that \( T(\alpha) = abc \). By definition of \( \alpha_M \), \( M^{(\alpha_M)} = M^{(\alpha)} \), so that

\[
M^{(\alpha)}_{ab} > M^{(\alpha)}_{ac} + M^{(\alpha)}_{cb}.
\]

But then by Definition 1, \( M^{(\alpha+1)}_{ab} \) is \( M^{(\alpha)}_{ac} + M^{(\alpha)}_{cb} \), whence \( M^{(\alpha+1)}_{ab} \neq M^{(\alpha_M)}_{ab} \). This is a contradiction, which completes the proof. That \( M^{(\alpha_M)} = G \) now follows by Lemma 2.

The cardinality of \( \alpha_M \) in the previous theorem can be arbitrary, but only because nothing prevents one from defining the class function \( T \) in a “wasteful” way, e.g., by mapping all ordinals between two uncountable limit ordinals into one and the same triad \( abc \). We can still estimate the cardinality of \( \alpha_M \) under a reasonable, “economic” organization of \( T \). For this, however, we need an auxiliary construction.
Definition 5. Let $T : \text{Ord} \to \text{Tri}(\mathcal{S})$ be any class function as described at the beginning of this section. Define, for each ordinal $\alpha$, an ordinal $\iota_\alpha$ by transfinite recursion as follows:

1. $\iota_0 = 0$;
2. if $\alpha = \beta + 1$, then $\iota_\alpha$ is the least ordinal $> \iota_\beta$ such that for each $abc \in \text{Tri}(\mathcal{S})$ there exists $\iota_\beta \leq \gamma < \iota_\alpha$ with $T(\gamma) = abc$;
3. if $\alpha$ is a limit ordinal, then $\iota_\alpha = \sup_{\beta < \alpha} \iota_\beta$.

Lemma 6. If $M(\iota_\alpha) = M(\beta)$ for all $\iota_\alpha \leq \beta \leq \iota_{\alpha+1}$, then $M(\iota_\alpha) = G$.

Proof. If $M(\iota_\alpha) = M(\beta)$ for all $\iota_\alpha \leq \beta \leq \iota_{\alpha+1}$ then it must be that $M(\iota_\alpha)$ satisfies the triangle inequality. Otherwise, fix the least $\beta \geq \iota_\alpha$ such that $T(\beta) = abc$ for some $a, b, c \in \mathcal{S}$ with $M(\beta)ab > M(\beta)ac + M(\beta)cb$. By definition, we must have $\beta < \iota_{\alpha+1}$, so also $\beta + 1 \leq \iota_{\alpha+1}$. But then $M(\beta+1)ab$ is defined to be $M(\beta)ac + M(\beta)cb$, meaning that $M(\beta+1)ab \neq M(\beta)ab = M(\iota_\alpha)ab$, a contradiction. By Lemma 2, we must thus have $M(\iota_\alpha) = G$. □

We now use this result to estimate the cardinality of $\alpha_M$ in Theorem 4 under an “economic” organization of the class function $T$. Denoting $T([\iota_\alpha, \iota_{\alpha+1}]) = \{\beta : \beta < \iota_{\alpha+1}\}$, by Definition 5, $T([\iota_\alpha, \iota_{\alpha+1}]) = \text{Tri}(\mathcal{S})$ for any $\alpha$. The class function $T$ is “economic” if, for any $\alpha$, $T$ maps $[\iota_\alpha, \iota_{\alpha+1})$ onto $\text{Tri}(\mathcal{S})$ injectively. Such a class function can be constructed as follows. Fix a well-ordering $\preceq$ of $\text{Tri}(\mathcal{S})$. Define $T$ by transfinite recursion as follows. Fix an ordinal $\alpha$, and assume $T(\beta)$ is defined for each $\beta < \alpha$, with $T(\beta) \in \text{Tri}(\mathcal{S})$. We define $T(\alpha) \in \text{Tri}(\mathcal{S})$. Let $S_\alpha$ be the set of all $abc \in \text{Tri}(\mathcal{S})$ for which there is a $\beta < \alpha$ such that $T(\beta) \preceq abc$ and $T(\beta') \neq abc$ for all $\beta \leq \beta' < \alpha$. If $S_\alpha$ is nonempty, set $T(\alpha)$ equal to the $\preceq$-least element of $S_\alpha$. Otherwise, let $T(\alpha)$ be the $\preceq$-least element of $\text{Tri}(\mathcal{S})$.

Theorem 7. If the class function $T$ is such that for any ordinal $\alpha$, $T$ maps the interval $[\iota_\alpha, \iota_{\alpha+1})$ onto $\text{Tri}(\mathcal{S})$ injectively, then $|\alpha_M| \leq |\mathcal{S}|$.

Proof. Let $\alpha_0$ be the least ordinal such that $\iota_{\alpha_0} \geq \alpha_M$. From Lemma 6 it follows that for each $\alpha < \alpha_0$, there is some $\beta$ with $\iota_\alpha \leq \beta \leq \iota_{\alpha+1}$ such that $M(\alpha) \neq M(\beta)$. The least such $\beta$ cannot be a limit, since otherwise, for all $a, b \in \mathcal{S}$, we would have $M(\beta)ab = \inf_{\beta < \beta} M(\beta)ab = \inf_{\iota_\alpha \leq \beta' < \beta} M(\beta')ab = M(\iota_\alpha)ab$. The least such $\beta$ is therefore a successor, and we denote its predecessor by $\beta_\alpha$. It follows that $M(\beta_\alpha) \neq M(\beta_{\alpha+1})$, so that $\beta_\alpha$ belongs to the set $C$ of Lemma 3. This defines an injective function $\{\iota_\alpha : \alpha < \alpha_0\} \to C$, so by Lemma 3, $\{\iota_\alpha : \alpha < \alpha_0\}$ has cardinality $\leq |\mathcal{S}|$. Since $|[\iota_\alpha, \iota_{\alpha+1})| \leq |\text{Tri}(\mathcal{S})|$ by assumption, we have

$$|\alpha_M| \leq |\alpha_0| = \left| \bigcup_{\alpha < \alpha_0} [\iota_\alpha, \iota_{\alpha+1}) \right| = \left| \{\iota_\alpha : \alpha < \alpha_0\} \right| \cdot |\text{Tri}(\mathcal{S})| \leq |\mathcal{S}| \cdot |\text{Tri}(\mathcal{S})| = |\mathcal{S}| \cdot |\mathcal{S}| = |\mathcal{S}|.$$ 

This completes the proof. □

3. Conclusion

For both finite and infinite sets $\mathcal{S}$, finding the induced $p,q$-metric $G$ from a pre-dissimilarity $D$ is equivalent to correcting violations of the triangle inequality in all triads $abc$ with elements in $\mathcal{S}$. The corrections consist in replacing $Dab$ with $Dac + Dcb$ whenever the former exceeds the sum. These corrections can be done
sequentially, having enumerated the triads by ordinals (in the finite case, by natural numbers) in an arbitrary way, provided only that each triad is enumerated by unboundedly many ordinals (natural numbers). The latter means that every triad remains accessible after any position in the enumeration. In Dzhafarov (2010a) and the present paper it is shown that the corrections will stop at some step (at some natural number in the finite case, and at some infinite ordinal otherwise) because all violations of the triangle inequality at this step will have been corrected; the redefined pre-dissimilarity $D$ will then coincide with $G$. The well-known Floyd-Warshall algorithm for finding $\min_a X b$ for all elements $a, b$ and chains $X$ in a finite $\mathcal{S}$ shows that the sequence of triads can be organized so that the cardinality of the eventual step does not exceed the cardinality of $\text{Tri}(\mathcal{S})$. The same statement is shown in this paper to hold for infinite sets, where $|\text{Tri}(\mathcal{S})| = |\mathcal{S}|$.

**Appendix A. A Brief Account of Ordinals and Cardinals**

We recall here some basic facts about ordinals and cardinals. We refer the reader to any basic text on set theory for complete details, e.g., Halmos (1974). We work within Zermelo-Fraenkel set theory with the axiom of choice (ZFC). The basic objects we deal with are sets, in terms of which all our other terms (functions, relations, etc.) may be defined. A class is formally a formula $\varphi(x)$ in the language of ZFC (with parameters), and informally it is the “collection” $C$ of all $x$ that satisfy this formula. We abuse notation, and write $x \in C$ in place of $\varphi(x)$, and we say $C$ contains $x$. Not all classes are sets, and these are called proper classes. (The class of all sets, for example, is a proper class.) We can also define class functions to be formulas $\varphi(x, y)$ such that for every $x$ there is at most one $y$ satisfying $\varphi(x, y)$. Intuitively, a class function is a map $f$ from one class into another, and as a result we use the suggestive notation $f : C_1 \to C_2$, where $C_1$ is the class of all $x$ for which there exists a $y$ such that $\varphi(x, y)$ holds, and $C_2$ is any class containing all $y$ such that $\varphi(x, y)$ holds for some $x$ in the class $C_1$. In this case, for $x \in C_1$ we write $f(x)$ to indicate the unique $y \in C_2$ such that $\varphi(x, y)$ holds.

An ordinal is a set $\alpha$ such that each $\beta \in \alpha$ is a set and $\beta \subseteq \alpha$. Thus, for example, $\emptyset, \{\emptyset\}, \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$ are all ordinals. It can be shown that for any two ordinals $\alpha$ and $\beta$, either $\alpha = \beta$, $\alpha \in \beta$, or $\beta \in \alpha$. We write $\alpha \leq \beta$ to denote $\alpha = \beta$ or $\alpha \in \beta$; we write $\alpha < \beta$ to denote $\alpha \in \beta$. In the above example, we thus have $\emptyset < \{\emptyset\} < \emptyset, \{\emptyset\} < \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} < \cdots$.

The class of all ordinals is not a set, and thus $\leq$ is not a set relation. But $\leq$ can be thought of as well-ordering the class of ordinals in the following sense: if $S$ is any set of ordinals, then $\leq$ restricted to the elements of $S$ is a well-ordering of $S$. Thus, in particular, each ordinal is well-ordered by $\leq$. Conversely, every well-ordered set can be mapped by an order-preserving bijection onto some ordinal. The ordinals are therefore canonical representatives of all possible well-order types, i.e., isomorphism classes of well-ordered sets.

For each ordinal $\alpha$, $\alpha \cup \{\alpha\}$ is also an ordinal, and we call it the successor of $\alpha$ and denote it by $\alpha + 1$. If $\alpha$ is the successor of some ordinal, we call $\alpha$ a successor ordinal; otherwise, we call $\alpha$ a limit ordinal. If we identify $\emptyset$ with 0, and, having identified the natural number $n \geq 0$ with an ordinal, identify $n + 1$ with $n \cup \{n\},$
then the above displayed sequence becomes

\[0, 1, 2, 3, \ldots\]

and \(\leq\) coincides with the standard ordering of the natural numbers. Here, 0 is not a successor ordinal, but each of 1, 2, 3, \ldots are. The ordinal

\[\{0, 1, 2, 3, \ldots\}\]

is denoted \(\omega\), and it is the \((\leq)\)least infinite ordinal, and the least limit ordinal after 0.

An ordinal \(\alpha\) is a **cardinal** if it cannot be put into one-to-one correspondence with any ordinal \(\beta < \alpha\). Thus, for example, each of 0, 1, 2, \ldots are cardinals (called the **finite cardinals**), as is \(\omega\). The ordinal \(\omega + 1\), on the other hand, is not a cardinal, since it can be put into one-to-one correspondence with \(\omega < \omega + 1\). The **cardinality** of a set \(S\), denoted \(|S|\), is the least ordinal \(S\) can be bijected with; that such an ordinal always exists follows by the axiom of choice, since it implies that \(S\) can be well-ordered. Thus, for example, \(|\omega + 1| = |\omega|\) and \(|\mathbb{Q}| = \omega\). It is clear that the cardinality of a set is a cardinal.

Systems of arithmetic can be developed on each of the class of ordinals and class of cardinals. These satisfy many familiar properties from arithmetic on the natural numbers, but also many properties which the naturals do not have. For the purposes of our work here, the only relevant properties are that if \(\kappa\) and \(\lambda\) are cardinals at least one of which is infinite then \(\kappa \cdot \lambda = \max\{\kappa, \lambda\}\) (in particular, if \(S\) is any infinite set, then \(|S| \cdot \omega = |S|\)), and that if \(\beta\) is an ordinal and \(\{A_\alpha : \alpha < \beta\}\) is a collection of disjoint sets each of cardinality \(\leq \kappa\), then

\[\left|\bigcup\{A_\alpha : \alpha < \beta\}\right| \leq |\beta| \cdot \kappa.\]

Theorems involving ordinals are often proved by **transfinite induction**, which asserts that if some class contains an ordinal \(\alpha\) whenever it contains all ordinals \(\beta < \alpha\), then the class contains all ordinals. Similarly, we can define a class function on the ordinals by means of **transfinite recursion**: if, having defined it on all \(\beta < \alpha\), we can use our definition to define it on \(\alpha\), then we define it on all ordinals. Within the induction or recursion procedure, it is sometimes more convenient to handle separately the case \(\alpha = 0\), \(\alpha\) a successor ordinal, and \(\alpha\) a limit ordinal.

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