This is an open book take-home exam. You may use any books, notes, websites or other printed material that you wish but do not consult with any other person. Put your name on all papers submitted and please show all of your work so that I can see your reasoning. The six questions will be equally weighted in the grading. Return the completed exams by 5 PM Wednesday May 5, to my mailbox in the department office, under my office door MSB408, or by email.

1. Let \( \mathcal{F}(t) \) be the filtration generated by a Brownian motion \( W(t) \) and assume the existence of a risk-neutral measure \( \tilde{\mathbb{P}} \) and a risk-free discount process \( D(t) \) and interest rate process \( R(t) \) with \( dD(t) = R(t)D(t)dt \). Let \( V \) be an \( \mathcal{F}(T) \)-measurable random variable with \( V > 0 \) almost surely. Show that there is a generalized geometric Brownian motion \( V(t) \) adapted to \( \mathcal{F}(t) \) with \( V(T) = V \). In other words, all strictly positive payoffs come from generalized geometric Brownian motions.

2. In the situation of question 1, assume that the risk-neutral measure \( \tilde{\mathbb{P}} \) is equal to \( \tilde{\mathbb{P}}_{S_1} \), the risk-neutral measure derived from some generalized geometric Brownian motion \( S_1(t) \) such that \( D(t)S_1(t) \) is a \( \tilde{\mathbb{P}}_{S_1} \)-martingale. Let \( \tilde{\mathbb{P}}_{S_2} \) be the risk-neutral measure derived from some other generalized geometric Brownian motion \( S_2(t) \) so that \( D(t)S_2(t) \) is a \( \tilde{\mathbb{P}}_{S_2} \)-martingale. Show that \( \tilde{\mathbb{P}}_{S_1}(A) = \tilde{\mathbb{P}}_{S_2}(A) \) for all sets \( A \in \mathcal{F} \). In other words, the risk-neutral measure is unique.

3. If \( S(t) = S(0)e^{\int_0^t \nu(u)du + \int_0^t \sigma(u)dW(u)} \) is a martingale, what is the relationship between \( \nu(t) \) and \( \sigma(t) \)? Prove it.

4. If \( M(t) = \int_0^t h(u)dN(u) \) where \( N(t) \) is the semi-martingale with \( dN(t) = \alpha(t)dt + \beta(t)dW(t) \), \( g(t) \) is an adapted process, and \( [M,M](t) \) is the quadratic variation of \( M(t) \), what is the simplest expression for \( \int_0^t g(u)d[M,M](u) \) in terms of \( g(t), h(t), \alpha(t), \beta(t), \) and \( W(t) \)?

5. Use stochastic calculus to derive a formula for \( \mathbb{E}[W(t)] \) assuming only that you know that \( dW(t)dW(t) = dt \) and the usual rules of stochastic calculus.

6. If \( B_1(t) \) and \( B_2(t) \) are Brownian motions with \( dB_1(t)dB_2(t) = \rho(t)dt \) for an adapted process \(-1 \leq \rho(t) \leq 1 \) find an adapted process \( W(t) \) that is a Brownian motion independent of \( B_1(t) \).