Structure of the CAPM Covariance Matrix

James Bridgeman
University of Connecticut

Actuarial Research Conference - University of California at Santa Barbara

July 14, 2014
SEE HOW CAPM COVARIANCE MATRIX:
- ACTUALLY HOLDS THE EXPECTED RETURNS
- ACTIVELY REFLECTS THE MARKET WEIGHTS

MOSTLY PEDAGOGY

A TINY BIT OF LIGHT ON LITERATURE
CAPM SET-UP

- MARKET COVARIANCE MATRIX

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn}
\end{pmatrix}
\]

invertible, $\sigma_{ij} = \sigma_{ji}$, $\sigma_{ii} > 0$, and $\sigma_{ii}\sigma_{jj} > \sigma_{ij}^2$

- MARKET WEIGHTS & EXPECTED RETURNS

\[
w = \begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{pmatrix}
\]

market weights, $e = \begin{pmatrix}1 \\ \vdots \end{pmatrix}$, $e^T \cdot w = 1$, $w_i > 0$,

\[
r = \begin{pmatrix}
r_1 \\
r_2 \\
\vdots \\
r_n
\end{pmatrix}
\]

expected returns, $r_f$ risk-free rate, $(r - r_f e)$ market risk premia
ASSUMING MARKET IS EFFICIENT (I.E. IS A FRONTIER PORTFOLIO)

- COVARIANCE-RETURNS-WEIGHTS RELATIONSHIP

\[
(r - r_f e) = \Sigma \cdot w
\]

\[
w = \frac{\Sigma^{-1} \cdot (r - r_f e)}{e^T \cdot \Sigma^{-1} \cdot (r - r_f e)} \quad \text{(denominator ensures } e^T \cdot w = 1)\]

- RETURNS-BETAS RELATIONSHIP

\[
(r - r_f e) = \beta \left( w^T \cdot r - r_f \right) = \beta (r_M - r_f)
\]

where \( \beta = \frac{\Sigma \cdot w}{w^T \cdot \Sigma \cdot w} \)

and \( r_M = w^T \cdot r \) is the expected market return.
EXAM QUESTION

Be Sure To Emphasize Effect Of Negative Covariance

\[
\Sigma = \begin{pmatrix}
.01 & .10 & -.20 \\
.10 & .04 & .25 \\
-.20 & .25 & .09
\end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix}
.02 \\
.10 \\
.20
\end{pmatrix}, \quad r_f = .03, \text{ what is } \mathbf{w}?
\]

WAIT UNTIL GRADING TO WORK OUT THE ANSWER

ANSWER

\[
\Sigma^{-1} = \begin{pmatrix}
4.5 & 4.5 & -2.5 \\
4.5 & 3.0 & 1.7 \\
-2.5 & 1.7 & .73
\end{pmatrix}, \quad \mathbf{w} = \frac{\Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} = \begin{pmatrix}
-.27 \\
.80 \\
.47
\end{pmatrix}, \text{ Rats!}
\]

WHAT WENT WRONG?

AHA! Forgot Requirement \( \sigma_{ii} \sigma_{jj} > \sigma_{ij}^2 \). Easy To Fix Next Semester.
ENLIGHTENED PERHAPS, BUT STILL LAZY

EXAM TIME THE FOLLOWING SEMESTER
- Still Be Sure To Emphasize Effect Of Negative Covariance

\[ \Sigma = \begin{pmatrix} .010 & .015 & -0.028 \\ .015 & .040 & .05 \\ -0.028 & .05 & .090 \end{pmatrix}, \quad r = \begin{pmatrix} .02 \\ .01 \\ .015 \end{pmatrix}, \quad r_f = .03, \text{ what is } w? \]

STILL WAIT UNTIL GRADING TO WORK OUT THE ANSWER

ANSWER

\[ \Sigma^{-1} = \begin{pmatrix} -13 & 33 & -23 \\ 33 & -1.4 & 11 \\ -23 & 11 & -2.1 \end{pmatrix}, \quad w = \frac{\Sigma^{-1} \cdot (r - r_f e)}{e^T \cdot \Sigma^{-1} \cdot (r - r_f e)} = \begin{pmatrix} -1.92 \\ 2.03 \\ .89 \end{pmatrix}, \text{ Worse!} \]

WHAT WENT WRONG THIS TIME?
- God Knows, Just Avoid The Negative Covariances Next Semester
THIRD SEMESTER’S A CHARM?

Let Everything Be Positive, Including All Of The Risk Premia

\[
\Sigma = \begin{pmatrix}
.010 & .015 & .028 \\
.015 & .040 & .05 \\
.028 & .05 & .090
\end{pmatrix}, \quad r = \begin{pmatrix}
.04 \\
.10 \\
.20
\end{pmatrix}, \quad r_f = .03, \text{ what is } w?
\]

CONFIDENTLY WAIT TO WORK OUT THE ANSWER

ANSWER

\[
\Sigma^{-1} = \begin{pmatrix}
791 & 36 & -266 \\
36 & 83.5 & -57.6 \\
-266 & -57.6 & 126
\end{pmatrix}, \quad w = \frac{\Sigma^{-1} \cdot (r - r_f e)}{e^T \cdot \Sigma^{-1} \cdot (r - r_f e)} = \begin{pmatrix}
1.47 \\
.15 \\
-.62
\end{pmatrix}, \text{ No!}
\]

THERE’S MORE TO THIS THAN MEETS THE EYE!

WHAT DOES \( \Sigma \) HAVE TO LOOK LIKE TO BE A MARKET?
A FAMILIAR PHENOMENON IN THE LITERATURE

- BEST & GRAUER (1985, 1992)
  - Efficient Markets with all $w_i > 0$ form a segment of the Frontier with length $\rightarrow 0$ as number of assets $n \rightarrow \infty$

- Brennan & Lo (2010)
  - For any given $(r - r_f e)$, Impossible $\Sigma$ are those with some $w_i < 0$. Then $\mathbb{P}[\Sigma \text{ Impossible}] \uparrow$ geometrically with $n$ for reasonable distribution assumptions on $\Sigma$.

  - Introduce statistical Shrinkage techniques to transform Empirical $\Sigma$ into a Frontier Portfolio

- LEVY & ROLL (2010)
  - Empirical $\Sigma$ has "high" $\mathbb{P}$ of being "close" to a Frontier Portfolio, with "close" attained by $\pm \epsilon$ on $\sigma_{ii}$ and $r_i$. Note $\rho_{ij}$ can remain fixed.

- BOYLE (2012, 2014)
  - On large class of $\Sigma$, Frontier Portfolio equivalent to $\Sigma$ Almost Positive
IS THERE A SIMPLE-MINDED WAY TO SEE ALL THIS?

- GIVEN $\mathbf{w}$ and $\mathbf{r} - r_f \mathbf{e}$ WITH ALL $w_i > 0$
- WHAT DO $\Sigma$ THAT SIT ON FRONTIER LOOK LIKE?

$$
\begin{align*}
\mathbf{w} &= \frac{\Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \quad \text{so} \\
\Sigma \cdot \mathbf{w} &= \frac{(\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \quad \text{and for any } k \\
(k \Sigma) \cdot \mathbf{w} &= \frac{k (\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})} \\
&= \frac{(\mathbf{r} - r_f \mathbf{e})}{\mathbf{e}^T \cdot (k \Sigma)^{-1} \cdot (\mathbf{r} - r_f \mathbf{e})}
\end{align*}
$$

So it is enough to find all $\Sigma$ such that

$$
\Sigma \cdot \mathbf{w} = (\mathbf{r} - r_f \mathbf{e})
$$

and then multiply by any constant.
IS THERE A SIMPLE-MINDED WAY TO SEE ALL THIS?

• EASY TO FIND A MATRIX $\mathbf{R}$ WITH $\mathbf{R} \cdot \mathbf{w} = \mathbf{r} - r\mathbf{f e}$

    Just let $\mathbf{R} = \left\{ (\mathbf{r} - r\mathbf{f e}) \mathbf{e}^T \right\}$

    with $n$ identical column vectors each equal to $(\mathbf{r} - r\mathbf{f e})$. We can also express $\mathbf{R}$ as $n$ row vectors

    $\mathbf{R} = \left\{ \begin{bmatrix} (r_1 - r_f) \mathbf{e}^T \\ (r_2 - r_f) \mathbf{e}^T \\ \vdots \\ (r_n - r_f) \mathbf{e}^T \end{bmatrix} \right\}$.

• UNFORTUNATELY THE CHOICE $\Sigma = \mathbf{R}$ IS NOT INVERTIBLE

    • Rows and columns clearly fail to be independent

    • COULD EASILY FAIL TO HAVE $\sigma_{ii} > 0$

    • IS UNLIKELY TO HAVE $\sigma_{ij} = \sigma_{ji}$ OR $\sigma_{ii} \sigma_{jj} > \sigma_{ij}^2$ FOR ALL $i, j$
BUT MAYBE THERE IS A MATRIX \( \mathbf{A} \) WITH \( \mathbf{A} \cdot \mathbf{w} = 0 \)

\[
\text{So } (\mathbf{R} + \mathbf{A}) \cdot \mathbf{w} = \mathbf{R} \cdot \mathbf{w} = (\mathbf{r} - r_f \mathbf{e})
\]

AND WITH \( \Sigma = (\mathbf{R} + \mathbf{A}) \) INVERTIBLE AND SATISFYING

\[
\sigma_{ij} = \sigma_{ji}, \quad \sigma_{ii} > 0, \quad \text{and } \sigma_{ii} \sigma_{jj} > \sigma_{ij}^2
\]

VIEW \( \mathbf{A} \) AS ROW VECTORS

\[
\mathbf{A} = \begin{pmatrix}
\mathbf{a}_1^T \\
\mathbf{a}_2^T \\
\vdots \\
\mathbf{a}_n^T
\end{pmatrix}, \text{ with each } \mathbf{a}_i^T = \{ a_{i1}, a_{i2}, \ldots, a_{in} \}
\]

THE CONDITION FOR \( \mathbf{A} \cdot \mathbf{w} = 0 \) IS

\[\mathbf{a}_i^T \cdot \mathbf{w} = 0 \text{ for all } i\]

in other words, all \( n \) of the row vectors \( \mathbf{a}_i^T \) must be in the \((n - 1)\)-dimensional hyperplane through the origin perpendicular to the market weight vector \( \mathbf{w} \).
IF ALL $\Sigma$ ARISE THIS WAY
AND IF ALL OF THE OTHER CONDITIONS CAN BE MET
(Perhaps two big IFs)
THEN

The odds of an $(n - 1)$-dimensional hyperplane through the origin being perpendicular to a vector $w$ with all $w_i > 0$ is $\left(\frac{1}{2}\right)^{n-1}$

- Half the lines through the origin in 2-space are perpendicular to something in first quadrant.
- A quarter of the planes through the origin in 3-space are perpendicular to something in the first octant.
- And so on ...

The lazy pedagogue had at best 1 chance in 4 to write down a valid $\Sigma$
even after he respected all of the $\sigma$ constraints

It’s immediately clear that the odds to randomly encounter a valid $\Sigma$ disappear at least exponentially in $n$, independent of (reasonable) distribution assumptions.
\( \sigma_{ij} = \sigma_{ji} \) means that
\[
(r_i - r_f) + a_{ij} = (r_j - r_f) + a_{ji}
\]
so \( a_{ij} = a_{ji} - (r_i - r_j) \)

and \( A \) can be expressed also as consisting of \( n \) column vectors
\[
A = \{ c_1 \ c_2 \ \ldots \ c_n \} \text{ where } c_j = a_j - (r - r_f e).
\]

\( \sigma_{ii} > 0 \) means that
\[
(r_i - r_f) + a_{ii} > 0
\]
so \( a_{ii} > -(r_i - r_f) \).

\( \sigma_{ii}\sigma_{jj} > \sigma_{ij}^2 \) means that
\[
(a_{ii} + (r_i - r_f))(a_{jj} + (r_j - r_f)) > (a_{ij} + (r_i - r_f))^2
\]
so \( a_{jj} > -(r_j - r_f) + \frac{(a_{ij} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)} \).

Looks like an induction might be possible.
\[ \Sigma = R + A = \left\{ \begin{array}{l}
(r_1 - r_f) e^T + a_1^T \\
(r_2 - r_f) e^T + a_2^T \\
\vdots \\
(r_n - r_f) e^T + a_n^T
\end{array} \right\} \]

IS INVERTIBLE IFF ITS \( n \) ROW VECTORS ARE INDEPENDENT

- INDEPENDENCE HOLDS IF AND ONLY IF
  1. The \( n \) row vectors \( a_1^T, a_2^T, \ldots, a_n^T \) span the \((n - 1)\)-dimensional hyperplane perpendicular to the market weight vector \( w \), and
  2. \( w^T \cdot (r - r_f e) \neq 0 \), i.e. the market risk premium vector \((r - r_f e)\) is not perpendicular to the market weight vector \( w \).

- PROOF
  1. \((r_1 - r_f) e^T, \ldots, (r_n - r_f) e^T\) are collinear so the row vectors of \( R + A \) cannot span an \( n \)-space unless \( a_1^T, \ldots, a_n^T \) span an \((n - 1)\)-space, and the \((n - 1)\)-dimensional hyperplane perpendicular to the market weight vector \( w \) is the \((n - 1)\)-space they are in.
PROOF (continued)

1. (prior slide)
2. Requires a slightly fussy proof that essentially follows from the fact that \( e^T \) and \( w \) both have all components positive and \( a_1^T, \ldots, a_n^T \) are all perpendicular to \( w \).

CONDITION 2 HOLDS IN ANY REASONABLE MODEL

\[
\begin{align*}
    w^T \cdot (r - r_f e) &= (w^T \cdot r - r_f) = (r_M - r_f) > 0
\end{align*}
\]

where \( r_M = w^T \cdot r \) is the expected return on the risky market.

SO \( \Sigma = R + A \) IS INVERTIBLE IN A REASONABLE MODEL IF AND ONLY IF

1. The \( n \) row vectors \( a_1^T, a_2^T, \ldots, a_n^T \) span the \( (n - 1) \)-dimensional hyperplane perpendicular to the market weight vector \( w \).
HOW HARD IS IT TO CHOOSE $\mathbf{a}$?

WORK BACKWARDS TO CHOOSE $\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_n^T$

- Suppose you already have chosen $\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_{n-1}^T$.
- Then there is no choice about what $\mathbf{a}_n^T$ must be. By symmetry:

  $a_{n1} = a_{1n} - (r_n - r_1)$
  $a_{n2} = a_{2n} - (r_n - r_2)$
  ...  
  $a_{nn-1} = a_{n-1}n - (r_n - r_{n-1})$.

Since $\mathbf{a}_n^T \cdot \mathbf{w} = 0$ the choice for $a_{nn}$ also is fixed. The requirement is

$$w_1 a_{n1} + \ldots + w_{n-1} a_{nn-1} + w_n a_{nn} = 0$$

so $a_{nn} = -\frac{1}{w_n} (w_1 a_{n1} + \ldots + w_{n-1} a_{nn-1})$,

which finishes the complete determination of $\mathbf{a}_n^T$. 

(Actuarial Research Conference - University of California at Santa Barbara)
HOW HARD IS IT TO CHOOSE $A$?

- **WORK BACKWARDS TO CHOOSE** $a_1^T, a_2^T, \ldots, a_n^T$
  
  But $a_{nn}$ has to satisfy some $\sigma$ conditions:
  
  $$a_{nn} > -(r_n - r_f) \text{ and for all } i < n$$
  
  $$a_{nn} > -(r_n - r_f) + \frac{(a_{in} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)}$$
  
  That means the choice of $a_{n-1}^T$ couldn’t have been completely free
  
  $$-\frac{1}{w_n} (w_1 a_{n1} + \ldots + w_{n-1} a_{n\, n-1}) > -(r_n - r_f) \text{ and for all } i < n$$
  
  $$-\frac{1}{w_n} (w_1 a_{n1} + \ldots + w_{n-1} a_{n\, n-1}) > -(r_n - r_f) + \frac{(a_{in} + (r_i - r_f))^2}{a_{ii} + (r_i - r_f)}$$
  
  where $a_{n1} = a_{1\, n} - (r_n - r_1)$
  
  $$\ldots$$
  
  $$a_{n\, n-1} = a_{n-1\, n} - (r_n - r_{n-1})$$
  
  Since all $w_i > 0$, just pick a small enough $a_{n-1\, n}$ (negative if need be)
HOW HARD IS IT TO CHOOSE $A$?

**WORK BACKWARDS TO CHOOSE $a_1^T$, $a_2^T$, ... , $a_n^T$**

- One possible problem at $i = n - 1$:

  $$\frac{1}{w_n} (w_1 a_n 1 + ... + w_{n-1} (a_{n-1} n - (r_n - r_{n-1}))) >$$

  $$> - (r_n - r_f) + \frac{(a_{n-1} n + (r_{n-1} - r_f))^2}{a_{n-1} n_{-1} + (r_{n-1} - r_f)}$$

  so $a_{n-1} n$ on both sides, and squared (so made positive) on the right. Does picking $a_{n-1} n$ small enough (negative if need be) still work?

- **YES!** $a_{n-1}^T$ is perpendicular to $w$ with all $w_j > 0$ so picking a small enough $a_{n-1} n$ (negative if need be) forces $a_{n-1} n_{-1}$ to increase and it turns out (some delicate analysis) to be enough to make the inequality work.
HOW HARD IS IT TO CHOOSE A?

THE HARD PART IS DONE

- Given $a_1^T, a_2^T, \ldots, a_{n-2}^T$ we saw that we can choose $a_{n-1}^T, a_n^T$ that satisfy the $\sigma$ conditions.

GO BY INDUCTION STARTING AT $a_1^T$

- For $1 \leq i \leq n - 2$, given $a_1^T, a_2^T, \ldots, a_{i-1}^T$ always free to choose $a_{ii}$ big enough to satisfy the $\sigma$ conditions, then $a_{i+1}, \ldots, a_n$ any values that keep $a_i^T \cdot w = 0$

SOME SPAN RESTRICTIONS ON CHOICES OF $a_i^T$

- To ensure that $a_1^T, a_2^T, \ldots, a_i^T$ spans at least an $(i - 1)$-dimensional subspace of the $(n - 1)$-dimensional hyperplane perpendicular to $w$ we have to disallow some choices in the induction.
- The whole set of disallowed matrices $A$ altogether has dimension $\leq \frac{n(n-1)}{2} - 3$.
- The whole set of allowed matrices $A$ has dimension $\frac{n(n-1)}{2}$, still for a fixed choice of $w$ and $(r - r_f e)$ and still requiring $\Sigma \cdot w = (r - r_f e)$
A FEW MORE DEGREES OF FREEDOM

We can multiply any $\Sigma = R + A$ developed above by any constant $k$

We can choose $w_1, \ldots, w_n$ subject to $e^T \cdot w = 1$ and $w_i > 0$

So altogether the space of possible solutions $\Sigma$ has dimension

$$\frac{n(n-1)}{2} + 1 + n - 1 = \frac{(n+1)n}{2},$$

now with a possible disallowed set of dimension

$$\leq \frac{(n+1)n}{2} - 3.$$

$A \cdot w = 0$ IS THE MAIN NON-OBVIOUS RESTRICTION

The $\sigma$ restrictions are all visible in $\Sigma$ and true for any empirical $\Sigma$

Invertibility of $\Sigma$ is apparent or easy to check, at least for small $n$.

The space of possible $n \times n$ symmetric matrices has dimension

$$n + (n - 1) + \ldots + 1 = \frac{(n+1)n}{2},$$

same as the space of solutions.

The disallowed set of matrices has measure (probability) 0.

But the odds of an $(n - 1)$-dimensional hyperplane through the origin

being perpendicular to a vector $w$ with all $w_i > 0$ is

$$\left(\frac{1}{2}\right)^{n-1}$$

So solutions seemingly are rare only because of that factor

$$\left(\frac{1}{2}\right)^{n-1}$$
REMEMBER THAT RETURNS ARE CONSTRAINED

In meaningful models

\[ w^T (r - r_f e) = (w^T r - r_f) = (r_M - r_f) > 0 \]

If there are any risky assets with expected return \( r_i < r_f \) then this inequality cuts off a fraction (call it \( f \)) of the otherwise possible \( w \).

Now the only possible \( w \) are in the intersection of the set having all \( w_i > 0 \) with the half-space having \( w^T (r - r_f e) > 0 \).

This in turn eliminates the same fraction \( f \) of the set of otherwise possible matrices \( A \), whose row vectors have to live on the plane perpendicular to \( w \), so possible \( \Sigma \) are now rare by a factor \( f \left( \frac{1}{2} \right)^{n-1} \).

This further militates against the lazy pedagogue’s chance of success, since he likes to illustrate negative covariances and negative risk premia.

It impairs by the same factor the probability of a random empirical \( \Sigma \) to satisfy CAPM, even if it satisfies all the \( \sigma \) constraints and invertibility.

If I have understood Phelim Boyle’s work, the "almost positive" condition does not yet contemplate this possibility. How should it be modified/generalized to this case?
Claim: \( A = \text{Cov} \left( \bar{r}, \left( \bar{r}^T - \bar{r}_M e^T \right) \right) \) where bar means random,

\( \bar{r} \) is the random vector of actual asset returns in the market and

\( \bar{r}_M \) is the random actual return on the market as a whole, i.e.

\[ \bar{r}_M = \bar{r}^T \cdot w \]

Proof: \( \text{Cov} \left( \bar{r}, \bar{r}^T \right) = \Sigma \) by definition.

\[
\begin{align*}
\text{Cov} \left( \bar{r}, \bar{r}_M e^T \right) &= \left\{ \text{Cov} \left( \bar{r}, \bar{r}_M \right) e^T \right\}, \text{ with } n \text{ identical columns} \\
&= \left\{ \text{Cov} \left( \bar{r}, \left( \bar{r}^T \cdot w \right) \right) e^T \right\} \\
&= \left\{ \left( \text{Cov} \left( \bar{r}, \bar{r}^T \right) \cdot w \right) e^T \right\} \\
&= \left\{ \left( \Sigma \cdot w \right) e^T \right\} = \left\{ \left( r - r_f e \right) e^T \right\} = R \text{ so} \\
\text{Cov} \left( \bar{r}, \left( \bar{r}^T - \bar{r}_M e^T \right) \right) &= \Sigma - R = A
\end{align*}
\]
THANKS

PHELIM BOYLE

MY STUDENTS THIS SUMMER