Calibration of a Regime-Switching Interest Rate Model

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Context for the Model

- **Long-Rate Anchor:** 20 Yr, Not (yet) Whole Curve
- **Stress-testing**
  - Not Forecasting
  - Not Pricing
- **What’s Important:**
  - Severe but Plausible Extreme Scenarios
    - Plausible: in historical context
    - Severe: represent real stresses
    - Extreme: on both (all) tails
- **Much Less Important:**
  - Accuracy Around the Likely Scenarios
- **Completely Irrelevant:**
  - Risk Neutrality
  - Arbitrage Free
Summary

- **Typical Generators (e.g. AAA).....**
  - Gaussian-based volatility driver
  - A single mean reversion point (MRP)

- **.....Fail To Produce Historically Plausible Ranges of Results**
  - Unhistorical shape to the realized volatility
  - Tightly bunched paths versus historical ranges
  - MRP assumption largely drives the extreme paths

- **To Fix the Problems**
  - Use fat-tailed volatility driver
  - Randomize MRP to spread range of extreme paths

- **But This Introduces More Parameters**
  - Calibration becomes a real challenge
History of 20 Year US Treasury Rate

Plausible By Definition
Neither Early 80’s Nor Japan Are Remotely Plausible In AAA
AAA Extreme Paths Are Not Japan-Like Near-Term - But They Persist
Historical Frequency of 20 Year Rates

frequency of 20 year rates

- LOGNORMAL
- HISTORICAL DATA

20 year treasury rate (%)

Frequency of Occurrence

0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14 0.16 0.18

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

Bridgeman Xie Zhang (Actuarial Research C, Calibration, August 2, 2013)
Monthly log-change in 20 year risk free rate
High Kurtosis

Frequency of monthly log-change in 20 year rates

- Historical Data
- Mean Gaussian
Stochastic Volatility Helps, May Not Fully Pick Up The Tails

frequency of monthly log-change in 20 year rates

- HISTORICAL DATA
- AAA Average

Monthly change in logarithm of 20 year treasury rate vs frequency of occurrence
Missing Tails Are Significant

frequency of monthly log-change in 20 year rates

- HISTORICAL DATA
- AAA Average +/- 1 std dev

Monthly change in logarithm of 20 year treasury rate
Comparative Statistics: History vs AAA

Rate Levels and Spread as well as the Shape of the Realized Volatility Differ Significantly from History

<table>
<thead>
<tr>
<th>Rate = 20 Year Treasury</th>
<th>60 Year History</th>
<th>AAA Mean</th>
<th>AAA StdDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate Mean</td>
<td>.0635</td>
<td>.0410</td>
<td>.0081</td>
</tr>
<tr>
<td>Rate StdDev</td>
<td>.0266</td>
<td>.0117</td>
<td>.0058</td>
</tr>
<tr>
<td>Rate Kurtosis (normal=3)</td>
<td>3.53</td>
<td>3.02</td>
<td>1.29</td>
</tr>
<tr>
<td>Rate 6th-osis (normal=15)</td>
<td>21.5</td>
<td>17.7</td>
<td>26.1</td>
</tr>
<tr>
<td>Realized Volatility = ΔlnRate</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volatility StdDev</td>
<td>.0360</td>
<td>.0338</td>
<td>.0039</td>
</tr>
<tr>
<td>Volatility Kurtosis (normal=3)</td>
<td>10.9</td>
<td>5.3</td>
<td>1.6</td>
</tr>
<tr>
<td>Volatility 6th-osis (normal=15)</td>
<td>479</td>
<td>76</td>
<td>124</td>
</tr>
</tbody>
</table>
Consider A New Model

- **Traditional Models (including AAA)**

  \[
  \Delta \ln Rate_t = \\
  F \ast (\ln MRP - \ln Rate_{t-1}) + SlopeAdjustment + (1 - F) \ast \text{Gaussian}\Delta
  \]

- **Proposed New Model:**

  Regime-Switching with Random Regimes

  \[
  \Delta \ln Rate_i = \\
  F \ast (\ln MRP_t - \ln Rate_{t-1}) - DriftCompensation + (1 - F) \ast \text{DiWeibull}\Delta
  \]

  where

  \[
  MRP_t = MRP_{t-1}
  \]

  unless

  \[
  t - t_{\text{regime}} > \text{a random Gamma}(\alpha, \beta) \text{ variate.}
  \]

  In that case, the regime switches to a new, random MRP:

  \[
  MRP_t = \text{a random LogNormal variate, fixed until next regime-switch.}
  \]

  And the regime-switching clock restarts at \( t_{\text{regime}} = t \).

  (\text{a SlopeAdjustment can be included if desirable})
What Is A DiWeibull?

DiWeibull Is Like Laplace:
Laplace is symmetric Exponential, DiWeibull is symmetric Weibull

Very Heavy Tail
A Sample Path From the New Model (enti-MRP 4-53)
New Model Requires 8 Parameters

- 2 Parameters For The Regime Clock Random $\text{Gamma}(\alpha, \beta)$ Variate.
  - $\alpha = 7.1$ and $\beta = 1.14$ (in annualized units) follows from MLE applied to historical random MRP estimates derived by Least Square Error analysis versus historical rates
  - Average length of an interest rate regime is $\alpha \beta = 8$ Years plus 1 Month

- 1 Initial Value For The MRP
  - Least Square Error analysis versus historical rates gives
    - For 4-1953 start: init-MRP=2.36%
    - For 6-2013 start: init-MRP=2.04%

- This Leaves 5 Parameters To Be Determined
  - 2 Parameters For The Lognormal Random MRP
  - 2 Parameters For The $\text{DiWeibull} \Delta$ Volatility Driver
  - 1 Mean Reversion Strength Factor ($F$ in the formula)

- Choose The 5 Parameters To Best Align Comparative Statistics vs History
Rate Levels and Spread as well as the Shape of the Realized Volatility Now Align With History

**Rate = 20 Year Treasury**

<table>
<thead>
<tr>
<th></th>
<th>60 Year History</th>
<th>Model Mean</th>
<th>Model StdDev</th>
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<tr>
<td>Rate Mean</td>
<td>0.0635</td>
<td>0.0631</td>
<td>0.0126</td>
</tr>
<tr>
<td>Rate StdDev</td>
<td>0.0266</td>
<td>0.0268</td>
<td>0.0105</td>
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<tr>
<td>Rate Kurtosis (normal=3)</td>
<td>3.53</td>
<td>2.96</td>
<td>1.24</td>
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<tr>
<td>Rate 6th-osis (normal=15)</td>
<td>21.5</td>
<td>15.8</td>
<td>18.9</td>
</tr>
<tr>
<td>(6th Ctrl Mom/StdDev^6)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Realized Volatility = (\Delta \ln \text{Rate})</strong></td>
<td></td>
<td></td>
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<tr>
<td>Volatility StdDev</td>
<td>0.0360</td>
<td>0.0363</td>
<td>0.0027</td>
</tr>
<tr>
<td>Volatility Kurtosis (normal=3)</td>
<td>10.9</td>
<td><strong>10.9</strong></td>
<td>4.8</td>
</tr>
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<td>479</td>
<td><strong>365</strong></td>
<td>636</td>
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Only 55/723 Months Breach 5%-95%: History Fits Into This Easily
Fits Like A Glove
Too Far In The Other Direction? At Least The Tail Is Good
AAA Vs New Model (init-MRP 6-13): Rate Frequency

Same Prob. $\leq 2.25\%$, Wild Difference Thereafter
An Extreme Path In The New Model (init-MRP 6-13)

For First 15 Years Slightly More Stress Than The 99%-ile AAA Scenario (And After 15 It Has Different Stresses That AAA Would Never Generate)
Shape Of Model Realized Volatility Is Not Only Fatter-Tailed On Average But Also Much More Varied

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<tr>
<td>Rate Mean</td>
<td>0.0628</td>
<td>0.0126</td>
<td>0.0410</td>
<td>0.0081</td>
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<tr>
<td>Rate StdDev</td>
<td>0.0271</td>
<td>0.0104</td>
<td>0.0117</td>
<td>0.0058</td>
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<tr>
<td>Rate Kurtosis (normal=3)</td>
<td>2.94</td>
<td>1.19</td>
<td>3.02</td>
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(6th Ctrl Mom/StdDev^6)

**Realized Volatility = Δ lnRate**

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<td>0.0027</td>
<td>0.0338</td>
<td>0.0039</td>
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<td>Volatility Kurtosis (normal=3)</td>
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<td>76</td>
<td>124</td>
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Both Miss Parts of Historical Volatility Shape Despite Other Evidence
Instead of Kurtosis and 6th-osis:

- Minimize L2 Distance of CDF to History
\[ \sqrt{\int (F(r) - H(r))^2 \, dr} \]

- Minimize L1 Distance of CDF to History
\[ \int |F(r) - H(r)| \, dr \]

- Use CDF Rather Than PDF To Emphasize Tails
- Use Both Rates and Realized Volatility
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<td>.0058</td>
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<tr>
<td>L2 Distance to History</td>
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<td>.0271</td>
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<tr>
<td>L1 Distance to History</td>
<td>.0102</td>
<td>.0035</td>
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<td>.0018</td>
<td>.0338</td>
<td>.0039</td>
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<tr>
<td>L2 Distance to History</td>
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<tr>
<td>L1 Distance to History</td>
<td>.0027</td>
<td>.0004</td>
<td>.0031</td>
<td>.0013</td>
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Realized Vol. Comparison For This Alternative Calibration

Frequency of monthly log-change in 20 year rates

- AAA Average
- Model Average
- HISTORICAL DATA

Monthly change in logarithm of 20 year treasury rate

Frequency of occurrence
With This Calibration The Volatility Driver Has Milder Tail
BiModal Not A Problem: Mean-Reversion Smooths It Out In Realized Vol.
Rate Distr. Comparison For This Alternative Calibration

The graph shows the frequency of 20-year rates with data points for AAA Average and Model Average. The x-axis represents the 20-year treasury rate (%) ranging from 0 to 25, and the y-axis represents the frequency of occurrence ranging from 0 to 0.18.
Monthly %-iles vs History For This Alternative Calibration

20 year risk-free rate: History vs Model

- 20 yr rate
- 99.5 percentile
- 95 percentile
- Mean
- 5 percentile
- 0.5 percentile

Date Range: April 53 to April 13
And Compared To AAA Generator

![Graph showing 20 year risk-free rate: Model vs AAA & History](image-url)
Extreme Path In This Alternative Calibration

Still Japan-like For A Good 15 Years