Illustrations of a Regime-Switching Stochastic Interest Rate Model With Randomized Regimes

James G. Bridgeman, FSA

University of Connecticut

ARC August 14, 2008
Introduction

- Work in progress
Introduction

- Work in progress
- Two years ago:
Work in progress

Two years ago:

- Unconstrained lognormal interest rate models have too much tail
Introduction

- Work in progress
- Two years ago:
  - Unconstrained lognormal interest rate models have too much tail
  - Mean-reverting ones have too little shoulder
Introduction

- Work in progress
- Two years ago:
  - Unconstrained lognormal interest rate models have too much tail
  - Mean-reverting ones have too little shoulder
  - Randomizing the reversion target fixes it

---

Bridgeman (University of Connecticut)  Illustrations  ARC August 14, 2008
Introduction

- Work in progress

Two years ago:

- Unconstrained lognormal interest rate models have too much tail
- Mean-reverting ones have too little shoulder
- Randomizing the reversion target fixes it
- A desirable drift formula for the mean-reverting lognormal

This year:

- Some corrections/additions (see www.math.uconn.edu/~bridgeman)
- Numerical examples
Introduction

- Work in progress
- Two years ago:
  - Unconstrained lognormal interest rate models have too much tail
  - Mean-reverting ones have too little shoulder
  - Randomizing the reversion target fixes it
  - A desirable drift formula for the mean-reverting lognormal
  - Got stuck on drift formula with randomized target
Introduction

- Work in progress

Two years ago:
- Unconstrained lognormal interest rate models have too much tail
- Mean-reverting ones have too little shoulder
- Randomizing the reversion target fixes it
- A desirable drift formula for the mean-reverting lognormal
- Got stuck on drift formula with randomized target

Last year:
Introduction

- Work in progress

- Two years ago:
  - Unconstrained lognormal interest rate models have too much tail
  - Mean-reverting ones have too little shoulder
  - Randomizing the reversion target fixes it
  - A desirable drift formula for the mean-reverting lognormal
  - Got stuck on drift formula with randomized target

- Last year:
  - Drift formula with the randomized target
Introduction

- Work in progress

Two years ago:

- Unconstrained lognormal interest rate models have too much tail
- Mean-reverting ones have too little shoulder
- Randomizing the reversion target fixes it
- A desirable drift formula for the mean-reverting lognormal
- Got stuck on drift formula with randomized target

Last year:

- Drift formula with the randomized target
- Asymptotic to closed forms involving Laplace transforms
Introduction

- Work in progress

- Two years ago:
  - Unconstrained lognormal interest rate models have too much tail
  - Mean-reverting ones have too little shoulder
  - Randomizing the reversion target fixes it
  - A desirable drift formula for the mean-reverting lognormal
  - Got stuck on drift formula with randomized target

- Last year:
  - Drift formula with the randomized target
  - Asymptotic to closed forms involving Laplace transforms
  - Interesting probability results
Introduction

- Work in progress

Two years ago:
- Unconstrained lognormal interest rate models have too much tail
- Mean-reverting ones have too little shoulder
- Randomizing the reversion target fixes it
- A desirable drift formula for the mean-reverting lognormal
- Got stuck on drift formula with randomized target

Last year:
- Drift formula with the randomized target
- Asymptotic to closed forms involving Laplace transforms
- Interesting probability results

This year:
Introduction

- Work in progress

Two years ago:

- Unconstrained lognormal interest rate models have too much tail
- Mean-reverting ones have too little shoulder
- Randomizing the reversion target fixes it
- A desirable drift formula for the mean-reverting lognormal
- Got stuck on drift formula with randomized target

Last year:

- Drift formula with the randomized target
- Asymptotic to closed forms involving Laplace transforms
- Interesting probability results

This year:

- Some corrections/additions (see www.math.uconn.edu/~bridgeman)
Introduction

- Work in progress

- Two years ago:
  - Unconstrained lognormal interest rate models have too much tail
  - Mean-reverting ones have too little shoulder
  - Randomizing the reversion target fixes it
  - A desirable drift formula for the mean-reverting lognormal
  - Got stuck on drift formula with randomized target

- Last year:
  - Drift formula with the randomized target
  - Asymptotic to closed forms involving Laplace transforms
  - Interesting probability results

- This year:
  - Some corrections/additions (see www.math.uconn.edu/~bridgeman)
  - Numerical examples
Example: 56 Years of the 10-year Treasury Rate

10 year risk free rate 1953-2008 (monthly data)
The Distribution of those Interest Rates

![Graph showing frequency of 10-year rates with a lognormal distribution compared to the data points.](Image)
Lognormal 4th Moment Is Just Too High (6th too)
56 Years of Changes in the 10 Year Treasury Rate

Normalized absolute monthly log-change in 10 year risk-free rate
The Fix: Randomize the Reversion Target
The Fix: Randomize the Reversion Target
Lognormal Models

- Unconstrained:

\[ d\ln(r_t) = D_t \, dt + \sigma \, dW_t \]

- Mean-reverting:

\[ d\ln(r_t) = h_1 \, (1 + F) \, dt \left[ \ln(T_0) \, \ln(r_t) \, dt \right] + (1 + F) \, dt \, D_t \, dt + (1 + F) \, dt \, \sigma \, dW_t \]

actuarial folklore (circa 1970)

Black-Karasinski (1991)

With Randomized Reversion Target

\[ d\ln(r_t) = h_1 \, (1 + F) \, dt \left[ \ln(T_t) \, \ln(r_t) \, dt \right] + (1 + F) \, dt \, D_t \, dt + (1 + F) \, dt \, \sigma \, dW_t , \text{ where } \ln(T_t) \text{ is random} \]
Lognormal Models

- Unconstrained:
  \[ d \ln (r_t) = D_t \, dt + \sigma \sqrt{dt} \, N_t \]

- Mean-reverting:
  \[ d \ln (r_t) = h_1 \left( (1-F) \, \ln(T_t) - \ln(r_t) \right) \, dt + \sigma \, dW_t \]
Lognormal Models

- Unconstrained:
  - $d \ln(r_t) = D_t dt + \sigma \sqrt{dt} N_t$
  - $d \ln(r_t) = D_t dt + \sigma dW_t$

 actuarial folklore (circa 1970)

Black-Karasinski (1991) With Randomized Reversion Target
Lognormal Models

- **Unconstrained:**
  
  \[
  d \ln (r_t) = D_t \, dt + \sigma \sqrt{dt} N_t
  \]
  
  \[
  d \ln (r_t) = D_t \, dt + \sigma dW_t
  \]

- **Mean-reverting:**
Lognormal Models

- Unconstrained:
  \[ d \ln(r_t) = D_t \, dt + \sigma \sqrt{dt} \, N_t \]
  \[ d \ln(r_t) = D_t \, dt + \sigma dW_t \]

- Mean-reverting:
  \[ d \ln(r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_0) - \ln(r_{t-} - dt) \right] \]
  \[ + (1 - F)^{dt} \, D_t \, dt + (1 - F)^{dt} \, \sigma \sqrt{dt} \, N_t \]
  actuarial folklore (circa 1970)
Lognormal Models

- **Unconstrained:**
  
  \[ d \ln(r_t) = D_t \, dt + \sigma \sqrt{dt} \, \mathcal{N}_t \]
  \[ d \ln(r_t) = D_t \, dt + \sigma dW_t \]

- **Mean-reverting:**
  
  \[ d \ln(r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_0) - \ln(r_{t-dt}) \right] + (1 - F)^{dt} D_t \, dt + (1 - F)^{dt} \sigma \sqrt{dt} \, \mathcal{N}_t \]

  actuarial folklore (circa 1970)

  \[ d \ln(r_t) = \left\{ - \ln(1 - F) \left[ \ln(T_0) - \ln(r_t) \right] + D_t \right\} \, dt + \sigma dW_t \]

  Black-Karasinski (1991)
Lognormal Models

- **Unconstrained:**
  \[d \ln(r_t) = D_t \, dt + \sigma \sqrt{dt} \, \mathbb{N}_t\]
  \[d \ln(r_t) = D_t \, dt + \sigma \, dW_t\]

- **Mean-reverting:**
  \[d \ln(r_t) = \left[1 - (1 - F)^d_t\right] \left[\ln(T_0) - \ln(r_{t-dt})\right] + (1 - F)^d_t \, D_t \, dt + (1 - F)^d_t \, \sigma \sqrt{dt} \, \mathbb{N}_t\]
  actuarial folklore (circa 1970)
  \[d \ln(r_t) = \left\{- \ln(1 - F) \left[\ln(T_0) - \ln(r_t)\right] + D_t\right\} \, dt + \sigma \, dW_t\]
  Black-Karasinski (1991)

- **With Randomized Reversion Target**
Lognormal Models

- **Unconstrained:**
  \[ d \ln (r_t) = D_t \, dt + \sigma \sqrt{dt} N_t \]
  \[ d \ln (r_t) = D_t \, dt + \sigma dW_t \]

- **Mean-reverting:**
  \[ d \ln (r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_0) - \ln(r_{t-dt}) \right] \]
  \[ + (1 - F)^{dt} D_t \, dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \]
  Actuarial folklore (circa 1970)

- **Black-Karasinski (1991)**
  \[ d \ln (r_t) = \left\{ - \ln(1 - F) \left[ \ln(T) - \ln(r_t) \right] + D_t \right\} \, dt + \sigma dW_t \]

- **With Randomized Reversion Target**
  \[ d \ln (r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_t) - \ln(r_{t-dt}) \right] \]
  \[ + (1 - F)^{dt} D_t \, dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \]
  Where \( \ln(T_t) \) is random
Lognormal Models

- Mean-reverting:

\[
\begin{align*}
d \ln(r_t) &= h_1 \left(1 - F \right) dt + \left(1 - F \right) dt D_t dt + \left(1 - F \right) dt \sigma_p dt \\text{actuarial folklore (circa 1970)}\end{align*}
\]

- Black-Karasinski (1991)

With Randomized Reversion Target

\[
\begin{align*}
d \ln(r_t) &= h_1 \left(1 - F \right) dt \ln(T_0) \ln(r_t) dt + \left(1 - F \right) dt D_t dt + \left(1 - F \right) dt \sigma dW_t \end{align*}
\]

\[\text{where } \ln(T_t) \text{ is random}\]
Lognormal Models

- Mean-reverting:
  
  \[
  d \ln (r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln (T_0) - \ln (r_{t-dt}) \right] \\
  + (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \\
  \]

  actuarial folklore (circa 1970)
Lognormal Models

- Mean-reverting:
  \[ d \ln (r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_0) - \ln(r_{t-dt}) \right] + (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \]
  actuarial folklore (circa 1970)

- \[ d \ln(r_t) = \left\{ - \ln(1 - F) \left[ \ln(T_0) - \ln(r_t) \right] + D_t \right\} dt + \sigma dW_t \]
  Black-Karasinski (1991)
Lognormal Models

- Mean-reverting:
  
  \[
  d \ln (r_t) = \left[ 1 - (1 - F)^{dt} \right] [\ln(T_0) - \ln(r_{t-}dt)] \\
  + (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t 
  \]  
  actuarial folklore (circa 1970)

- \[
  d \ln(r_t) = \left\{ - \ln (1 - F) [\ln(T_0) - \ln(r_t)] + D_t \right\} dt + \sigma dW_t 
  \]  
  Black-Karasinski (1991)

- With Randomized Reversion Target
Lognormal Models

- **Mean-reverting:**
  
  \[ d \ln (r_t) = \left[ 1 - (1 - F)^{dt} \right] [\ln(T_0) - \ln(r_{t-dt})] \]
  
  \[ + (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \]

  Actuarial folklore (circa 1970)

- **Black-Karasinski (1991)**
  
  \[ d \ln(r_t) = \{- \ln(1 - F) [\ln(T_0) - \ln(r_t)] + D_t\} dt + \sigma dW_t \]

- **With Randomized Reversion Target**
  
  \[ d \ln(r_t) = \left[ 1 - (1 - F)^{dt} \right] [\ln(T_t) - \ln(r_{t-dt})] \]
  
  \[ + (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t, \text{ where } \ln(T_t) \text{ is random} \]
Lognormal Models

- **Mean-reverting:**
  
  \[ d \ln (r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_0) - \ln(r_{t-dt}) \right] + (1 - F)^{dt} D_t \ dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \]

  actuarial folklore (circa 1970)

- **With Randomized Reversion Target**

  \[ d \ln(r_t) = \left\{ - \ln(1 - F) \left[ \ln(T_0) - \ln(r_t) \right] + D_t \right\} dt + \sigma dW_t \]

  Black-Karasinski (1991)
The Target

- When does it (the target; the regime) switch? At randomly chosen times:

\[ t_{j+1} - t_j = \text{a set of i.i.d. random variables with common law gamma}(\alpha, \beta), \] the inter-arrival intervals for regime-switches.

- \( t_1 = \text{a random variable independent of } t_{j+1} - t_j \)

- Distributed as a randomly chosen point within a gamma(\( \alpha, \beta \)) interval.

What does it switch to? A lognormally distributed value:

\[ T_{j+1} = \text{i.i.d. lognormal mean reversion targets for the interest rate, thus characterizing each regime by a randomly chosen mean reversion target.} \]

\( T_t = T_{1+1}(1 - F dt)^{\sum_{j=0}^{k} T(1 - F dt)^{t_{j+1} - t_j}} \), where the product is over all \( t_j \) that fall in the interval \([t dt, t)\) plus the immediate prior.
The Target

- When does it (the target; the regime) switch? At randomly chosen times:

  \[ \{t_{j+1} - t_j\}_{1 \leq j} = \text{a set of i.i.d. random variables with common law } \gamma(\alpha, \beta), \text{ the inter-arrival intervals for regime-switches} \]
The Target

- When does it (the target; the regime) switch? At randomly chosen times:
  - \( \{t_{j+1} - t_j\}_{1 \leq j} \) = a set of i.i.d. random variables with common law \( \text{gamma}(\alpha, \beta) \), the inter-arrival intervals for regime-switches
  - \( t_1 \) = a random variable independent of \( \{t_{j+1} - t_j\}_{1 \leq j} \) distributed as a randomly chosen point within a \( \text{gamma}(\alpha, \beta) \) interval.
When does it (the target; the regime) switch? At randomly chosen times:

- \( \{t_{j+1} - t_j\}_{1 \leq j} \) = a set of i.i.d. random variables with common law gamma(\(\alpha, \beta\)), the inter-arrival intervals for regime-switches
- \( t_1 \) = a random variable independent of \( \{t_{j+1} - t_j\}_{1 \leq j} \) distributed as a randomly chosen point within a gamma(\(\alpha, \beta\)) interval.

What does it switch to? A lognormally distributed value:

\[ T_{j+1} = \text{i.i.d. lognormal mean reversion targets for the interest rate, thus characterizing each regime by a randomly chosen mean reversion target.} \]

(\( T_0 \) is a fixed target value for the first regime.)
The Target

- When does it (the target; the regime) switch? At randomly chosen times:
  - \( \{t_{j+1} - t_j\}_{1 \leq j} \) = a set of i.i.d. random variables with common law \( \text{gamma}(\alpha, \beta) \), the inter-arrival intervals for regime-switches
  - \( t_1 \) = a random variable independent of \( \{t_{j+1} - t_j\}_{1 \leq j} \) distributed as a randomly chosen point within a \( \text{gamma}(\alpha, \beta) \) interval.

- What does it switch to? A lognormally distributed value:
  - \( \{T_j\}_{1 \leq j} \) = i.i.d. lognormal mean reversion targets for the interest rate, thus characterizing each regime by a randomly chosen mean reversion target.
The Target

- When does it (the target; the regime) switch? At randomly chosen times:
  - \( \{ t_{j+1} - t_j \}_{1 \leq j} \) = a set of i.i.d. random variables with common law \( \text{gamma}(\alpha, \beta) \), the inter-arrival intervals for regime-switches
  - \( t_1 \) = a random variable independent of \( \{ t_{j+1} - t_j \}_{1 \leq j} \) distributed as a randomly chosen point within a \( \text{gamma}(\alpha, \beta) \) interval.
- What does it switch to? A lognormally distributed value:
  - \( \{ T_j \}_{1 \leq j} \) = i.i.d. lognormal mean reversion targets for the interest rate, thus characterizing each regime by a randomly chosen mean reversion target.
  - \( T_0 \) is a fixed target value for the first regime.)
The Target

- When does it (the target; the regime) switch? At randomly chosen times:
  - \( \{ t_{j+1} - t_j \}_{1 \leq j} \) = a set of i.i.d. random variables with common law \( \text{gamma}(\alpha, \beta) \), the inter-arrival intervals for regime-switches
  - \( t_1 \) = a random variable independent of \( \{ t_{j+1} - t_j \}_{1 \leq j} \) distributed as a randomly chosen point within a \( \text{gamma}(\alpha, \beta) \) interval.

- What does it switch to? A lognormally distributed value:
  - \( \{ T_j \}_{1 \leq j} \) = i.i.d. lognormal mean reversion targets for the interest rate, thus characterizing each regime by a randomly chosen mean reversion target.
  - \( T_0 \) is a fixed target value for the first regime.

\[
T_t = T_k \prod_{j=k'}^{k-1} \frac{(1-F)^{dt \wedge (t-t_j+1)} - (1-F)^{dt \wedge (t-t_j)}}{1-(1-F)^{dt}} \quad \text{, where the product is over all } t_j \text{ that fall in the interval } [t - dt, t) \text{ plus the immediate prior } t_j \text{ (see corrections to last year's paper at } \text{www.math.uconn.edu/~bridgeman)}
\]
Drift Compensation: the ordinary mean-reversion case

- It would be intuitive to have:

\[
\mathbb{E} [r_t] = r_0 (1-F)^t T_0 [1-(1-F)^t]
\]
Drift Compensation: the ordinary mean-reversion case

- It would be intuitive to have:
  \[ \mathbb{E} [r_t] = r_0 (1 - F)^t \frac{1 - (1 - F)^t}{T_0} \]

- To find out what drift \( D_t \) will ensure it, you can integrate \( d \ln(r_t) \):

\[
\ln(r_t) = \ln(r_0) (1 - F)^t \frac{dt}{dt} + \sigma \sqrt{dt} \sum_{s=1}^{t} N_{t-(s-1)dt} (1 - F)^{sdt} \\
+ \ln(T_0) \left[ 1 - (1 - F)^{dt} \right] \sum_{s=1}^{t} (1 - F)^{(s-1)dt} \quad \text{\( \leftarrow \text{notice geom. series} \)} \\
+ dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{sdt} \quad \text{which simplifies to:}
\]

\[
\ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t} N_{t-(s-1)dt} (1 - F)^{sdt} \\
+ \ln(T_0) \left[ 1 - (1 - F)^{dt} \right] + dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{sdt}, \text{ which is Gaussian.}
\]
Drift Compensation: the ordinary mean-reversion case

- Since $\ln(r_t)$ is Gaussian, $E[r_t] = e^{\mu} + \frac{1}{2} \sigma^2$ where the $\mu$ and $\sigma^2$ are some mess determined by the constants in the expression for $\ln(r_t)$. 

- Practical work with the randomized reversion target model all but requires you to know similar closed forms for drift compensation (can't do physical measure Monte Carlo without drift compensation), but now when you integrate, no convenient geometric series appears.

- Similarly for variance, which you need to calibrate a Monte Carlo.
Drift Compensation: the ordinary mean-reversion case

- Since $\ln(r_t)$ is Gaussian, $\mathbb{E} [r_t] = e^{\mu + \frac{1}{2} \sigma^2}$ where the $\mu$ and $\sigma^2$ are some mess determined by the constants in the expression for $\ln(r_t)$.

- If you work that mess out and set it equal to $r_0^0 (1-F)^T T_0^{1-(1-F)^T}$, and require that it be true for all $t$, you can arrive at what the drift compensation function $D_t$ must be to deliver the intuitive $\mathbb{E} [r_t]$:
Drift Compensation: the ordinary mean-reversion case

- Since $\ln(r_t)$ is Gaussian, $\mathbb{E}[r_t] = e^{\mu + \frac{1}{2} \sigma^2}$ where the $\mu$ and $\sigma^2$ are some mess determined by the constants in the expression for $\ln(r_t)$.

- If you work that mess out and set it equal to $r_0(1-F)^t T_0[1-(1-F)^t]$, and require that it be true for all $t$, you can arrive at what the drift compensation function $D_t$ must be to deliver the intuitive $\mathbb{E}[r_t]$:

$$D_t = -\frac{1}{2} \sigma^2 \frac{(1-F)^{dt}}{1+(1-F)^{dt}} \left[ 1 + (1 - F)^{2t-dt} \right]$$
Drift Compensation: the ordinary mean-reversion case

- Since $\ln(r_t)$ is Gaussian, $\mathbb{E}[r_t] = e^{\mu + \frac{1}{2} \sigma^2}$ where the $\mu$ and $\sigma^2$ are some mess determined by the constants in the expression for $\ln(r_t)$.

- If you work that mess out and set it equal to $r_0^{(1-F)^t} T_0^{[1-(1-F)^t]}$, and require that it be true for all $t$, you can arrive at what the drift compensation function $D_t$ must be to deliver the intuitive $\mathbb{E}[r_t]$:
  - $D_t = -\frac{1}{2} \sigma^2 \frac{(1-F)^t}{1+(1-F)^t} \left[ 1 + (1 - F)^{2t} - dt \right]$, or
  - $D_t = -\frac{1}{4} \sigma^2 \left[ 1 + (1 - F)^{2t} \right]$ in the continuous case.
Drift Compensation: the ordinary mean-reversion case

- Since \( \ln(r_t) \) is Gaussian, \( \mathbb{E}[r_t] = e^{\mu + \frac{1}{2} \sigma^2} \) where the \( \mu \) and \( \sigma^2 \) are some mess determined by the constants in the expression for \( \ln(r_t) \).

- If you work that mess out and set it equal to \( r_0^{(1-F)^t} T_0^{[1-(1-F)^t]} \), and require that it be true for all \( t \), you can arrive at what the drift compensation function \( D_t \) must be to deliver the intuitive \( \mathbb{E}[r_t] \):
  \[
  D_t = -\frac{1}{2} \sigma^2 \frac{(1-F)^t}{1+(1-F)^t} \left[ 1 + (1 - F)^{2t} - dt \right], \text{ or }
  \]

- \( D_t = -\frac{1}{4} \sigma^2 \left[ 1 + (1 - F)^{2t} \right] \) in the continuous case.

- There is a similar closed form for the variance of \( r_t \).
Drift Compensation: the ordinary mean-reversion case

- Since $\ln(r_t)$ is Gaussian, $\mathbb{E}[r_t] = e^{\mu + \frac{1}{2} \sigma^2}$ where the $\mu$ and $\sigma^2$ are some mess determined by the constants in the expression for $\ln(r_t)$.

- If you work that mess out and set it equal to $r_0 (1-F)^t T_0^{1-(1-F)^t}$, and require that it be true for all $t$, you can arrive at what the drift compensation function $D_t$ must be to deliver the intuitive $\mathbb{E}[r_t]$:

  - $D_t = -\frac{1}{2} \sigma^2 \frac{(1-F)^t}{1+(1-F)^t} \left[ 1 + (1 - F)^2 t - dt \right]$, or
  - $D_t = -\frac{1}{4} \sigma^2 \left[ 1 + (1 - F)^2 t \right]$ in the continuous case

- There is a similar closed form for the variance of $r_t$

- Practical work with the randomized reversion target model all but requires you to know similar closed forms for drift compensation (can’t do physical measure Monte Carlo without drift compensation), but now when you integrate, no convenient geometric series appears. Similarly for variance, which you need to calibrate a Monte Carlo.
In last year’s paper I showed that

\[ \ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t} N_{t-(s-1)dt} (1 - F)^{sdt} \]

\[ + \ln(T_0) \left[ (1 - F)^{t-t_1^+} - (1 - F)^t \right] \]

\[ + \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{t-t_{j+1}^+} - (1 - F)^{t-t_j^+} \right] \]

\[ + dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{sdt} \]

(1.3)
In last year’s paper I showed that

\[ \ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{\frac{t}{dt}} N_{t-(s-1)dt} (1 - F)^{sdt} \]

\[ + \ln(T_0) \left[ (1 - F)^{{t-t_1}+} - (1 - F)^t \right] \]

\[ + \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{{t-t_{j+1}}+} - (1 - F)^{{t-t_j}+} \right] \]

\[ + dt \sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt} (1 - F)^{sdt} \]

(1.3)

If you condition on \( \{t_j\} \) this is just Gaussian with parameters some mess determined by the coefficients of the \( N_{t-(s-1)dt} \) and the \( \ln(T_j) \).
Drift Compensation: with randomized reversion target

- In last year’s paper I showed that

$$\ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t} N_{t-(s-1)dt} (1 - F)^{sdt}$$

$$+ \ln(T_0) \left[ (1 - F)^{t-t_1} + (1 - F)^t \right]$$

$$+ \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{t-t_{j+1}} + (1 - F)^{t-t_j} \right]$$

$$+ dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{sdt}$$

(1.3)

- If you condition on \( \{t_j\} \) this is just Gaussian with parameters some mess determined by the coefficients of the \( N_{t-(s-1)dt} \) and the \( \ln(T_j) \).
- So conditional on \( \{t_j\} \) you can calculate the moments of \( r_t \) using knowledge of the moments of a lognormal.
But expressions involving $\ln(T_j) \left[ (1 - F)^{(t-t_{j+1})+} - (1 - F)^{(t-t_j)+} \right]$ make it a nightmare to find the unconditioned moments of $r_t$. 

\[ E_{h(r_t)} = e^{\mu + \frac{1}{2} \sigma^2 + \frac{1}{4} \sum_{j=2}^{\infty} \frac{\sum_{n=0}^{j-1} \frac{1}{n!} \frac{1}{(2n)!} \sum_{i=0}^{n} \frac{\sigma_i^2}{(2i)!}}{\sum_{n=0}^{j-1} \frac{1}{n!} \frac{1}{(2n)!} \sum_{i=0}^{n} \frac{\sigma_i^2}{(2i)!}} \} \]

where $\binom{2n}{n}$ is the central binomial coefficient and $\mu$, $\sigma^2$, and $\mu^j$ are the mean, variance, and higher central moments of $\ln(r_t)$. 

The first approximation (beyond the lognormal) is:

\[ E_{h(r_t)} = e^{\mu + \frac{1}{2} \sigma^2 + \frac{1}{4} \mu^4 + \frac{1}{16} \sigma^4} \]

So we derived an approximation series based on the old Edgeworth approximation.
Drift Compensation: with randomized reversion target

- But expressions involving \( \ln(T_j) \left[ (1 - F)^{(t-t_{j+1})_+} - (1 - F)^{(t-t_j)_+} \right] \) make it a nightmare to find the unconditioned moments of \( r_t \).
- So we derived an approximation series based on the old Edgeworth approximation:

\[
\mathbb{E} \left[ (r_t)^l \right] = e^{l\mu + \frac{1}{2}(l\sigma)^2} \left\{ 1 + \lim_{N \to \infty} \sum_{j=2}^{N} \frac{l^{2j}}{(2j)!} \left[ \mu_{2j} - (2j)!\sigma^{2j} \right] \cdot \sum_{n=0}^{N-j} \frac{(-1)^n (2n)!}{(2n)!} (l\sigma)^{2n} \right\}
\]

where \((2n)! = (2n - 1)(2n - 3) \cdots 1\) and \(\mu\), \(\sigma^2\), and \(\mu_{2j}\) are the mean, variance and higher central moments of \(\ln(r_t)\).
Drift Compensation: with randomized reversion target

- But expressions involving \( \ln(T_j) \left[ (1 - F)^{(t-t_{j+1})} - (1 - F)^{(t-t_j)} \right] \) make it a nightmare to find the unconditioned moments of \( r_t \).

- So we derived an approximation series based on the old Edgeworth approximation:

\[
E \left[ (r_t)^l \right] = e^{l\mu + \frac{1}{2} (l\sigma)^2} \left\{ 1 + \lim_{N \to \infty} \sum_{j=2}^{N} \frac{l^{2j}}{(2j)!} \left[ \mu_{2j} - (2j)! \sigma^{2j} \right] \right. \\
\left. \cdot \sum_{n=0}^{N-j} \frac{(-1)^n (2n)!}{(2n)!} (l\sigma)^{2n} \right\}
\]

where \((2n)! = (2n - 1)(2n - 3) \cdots 1\) and \( \mu, \sigma^2, \) and \( \mu_{2j} \) are the mean, variance and higher central moments of \( \ln(r_t) \).

- The first approximation (beyond the lognormal) is

\[
E \left[ (r_t)^l \right] \approx e^{l\mu + \frac{1}{2} (l\sigma)^2} \left\{ 1 + \frac{l^4}{4!} [\mu_4 - 3\sigma^4] \right\}
\]
Drift Compensation: with randomized reversion target

- The second approximation is (correcting an embarrassing error last year)

\[
\mathbb{E} \left[ (r_t)' \right] \approx e^{\mu + \frac{1}{2} (l\sigma)^2} \left\{ 1 + \frac{l^4}{4!} [\mu_4 - 3\sigma^4] \left( 1 - \frac{1}{2!} (l\sigma)^2 \right) \\
+ \frac{l^6}{6!} [\mu_6 - 15\sigma^6] \right\}
\]
The second approximation is (correcting an embarrassing error last year)

\[
\mathbb{E} \left[ (r_t)^I \right] \approx e^{\mu + \frac{1}{2} (l \sigma)^2} \left\{ 1 + \frac{l^4}{4!} [\mu_4 - 3 \sigma^4] \left( 1 - \frac{1}{2!} (l \sigma)^2 \right) + \frac{l^6}{6!} [\mu_6 - 15 \sigma^6] \right\}
\]

And the third is

\[
\mathbb{E} \left[ (r_t)^I \right] \approx e^{\mu + \frac{1}{2} (l \sigma)^2} \left\{ 1 + \frac{l^4}{4!} [\mu_4 - 3 \sigma^4] \left( 1 - \frac{1}{2!} (l \sigma)^2 \right) + \frac{3}{4!} (l \sigma)^4 \right\} + \frac{l^6}{6!} [\mu_6 - 15 \sigma^6] \left( 1 - \frac{1}{2!} (l \sigma)^2 \right) + \frac{l^8}{8!} [\mu_8 - 105 \sigma^6] \right\}
\]
Drift Compensation: with randomized reversion target

- Now we used that conditioning on \{t_j\} to find the mean and central moments of ln(r_t) needed here.

\[
\begin{align*}
\mu_T &= \ln(r_0) - \ln(T_0) + \ln(T_0) + \frac{1}{2} \sigma_T^2 \left(1 + E_h\left(1 + F\right)\right) - di + \mu_{T+1} + \frac{1}{2} \sigma_T^2 \left(1 + E_h\right)^2 - \sum_{j=1}^{L-1} e^{2j} \right) dt,
\end{align*}
\]

where (e.3) \(\mu_T\) and \(\sigma_T\) are the parameters for the lognormal \(f_T\) and \(d_i\) follows the equilibrium distribution of our gamma \((\alpha, \beta)\) interarrival times for regime-switches and \(E_h\left(1 + F\right)\) is a calculable Laplace transform.
Now we used that conditioning on \( \{ t_j \} \) to find the mean and central moments of \( \ln(r_t) \) needed here.

\[
\mu = \left[ \ln(r_0) - \ln(T_0) \right] (1 - F)^t + \left[ \ln(T_0) - \left( \mu_T + \frac{1}{2} \sigma_T^2 \right) \right] (1 - F)^t \mathbb{E} \left[ (1 - F)^{-t/d} \right]
\]

\[
+ \mu_T + \frac{1}{2} \sigma_T^2 \left[ 1 - \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right] \right]
\]

\[
- \frac{1}{2} \sigma^2 dt (1 - F)^{2dt} \frac{1 - (1 - F)^{2t}}{1 - (1 - F)^{2dt}}, \text{ where}
\]
Drift Compensation: with randomized reversion target

- Now we used that conditioning on \( \{t_j\} \) to find the mean and central moments of \( \ln(r_t) \) needed here.

\[
\mu = \left[ \ln(r_0) - \ln(T_0) \right] (1 - F)^t \\
+ \left[ \ln(T_0) - \left( \mu_T + \frac{1}{2} \sigma_T^2 \right) \right] (1 - F)^t \mathbb{E} \left[ (1 - F)^{-t/d} \right] \\
+ \mu_T + \frac{1}{2} \sigma_T^2 \left[ 1 - \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right] \right] \\
- \frac{1}{2} \sigma^2 dt (1 - F)^{2dt} \frac{1 - (1 - F)^{2t}}{1 - (1 - F)^{2dt}}, \text{ where} \quad (e.3)
\]

- \( \mu_T \) and \( \sigma_T \) are the parameters for the lognormal \( \{T_j\} \).
Now we used that conditioning on \( \{t_j\} \) to find the mean and central moments of \( \ln(r_t) \) needed here.

\[
\mu = \left[ \ln(r_0) - \ln(T_0) \right] (1 - F)^t + \left[ \ln(T_0) - \left( \mu_T + \frac{1}{2} \sigma_T^2 \right) \right] (1 - F)^t \mathbb{E} \left[ (1 - F)^{-t \wedge d} \right] \\
+ \mu_T + \frac{1}{2} \sigma_T^2 \left[ 1 - \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right] \right] \\
- \frac{1}{2} \sigma^2 dt (1 - F)^{2dt} \frac{1 - (1 - F)^{2t}}{1 - (1 - F)^{2dt}}, \text{ where (e.3)}
\]

\( \mu_T \) and \( \sigma_T \) are the parameters for the lognormal \( \{T_j\} \).

\( d \) follows the equilibrium distribution of our \( \text{gamma}(\alpha, \beta) \) interarrival times for regime-switches and \( \mathbb{E} \left[ (1 - F)^{-t \wedge d} \right] = \mathcal{L}_d \wedge t (\ln(1 - F)) \) is a calculable Laplace transform.
Drift Compensation: with randomized reversion target

\[ e_j = \left\{ (1 - F)^{t-t_{j+1}} - (1 - F)^{t-t_j} \right\}, \] and a delicate evaluation of \[ \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right] \] was the main burden of last year’s paper (and corrections.)
Drift Compensation: with randomized reversion target

- $e_j = \left\{ (1 - F)^{(t-t_{j+1})}+ - (1 - F)^{(t-t_j)} \right\}$, and a delicate evaluation of $\mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right]$ was the main burden of last year’s paper (and corrections.)

$$\mathbb{E} \left[ \{\ln(r_t) - \mathbb{E}[\ln(r_t)]\}^2 \right] =$$

$$= \sigma^2 dt (1 - F)^{2dt} \frac{1 - (1 - F)^{2t}}{1 - (1 - F)^{2dt}} + \sigma_T^2 \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right]$$

$$+ (\ln(T_0) - \mu_T)^2 (1 - F)^{2t} \left\{ \mathbb{E} \left[ (1 - F)^{-2t_1 \wedge t} \right] \right.$$

$$- \left. \left( \mathbb{E} \left[ (1 - F)^{-t_1 \wedge t} \right] \right)^2 \right\}$$

(2.6.c)

is the variance of $\ln(r_t)$
This allows calibration of the drift compensation term as

\[
D_t = -\frac{1}{2} \sigma^2 \frac{(1 - F)^{dt}}{1 + (1 - F)^{dt}} \left( 1 + (1 - F)^{2t - dt} \right) \\
+ \frac{1}{2} \sigma^2 T \frac{1}{dt (1 - F)^{dt}} \left\{ 1 - (1 - F)^{dt} \\
- (1 - F)^t \left( \mathbb{E} \left[ (1 - F)^{-t \wedge d} \right] - \mathbb{E} \left[ (1 - F)^{-(t - dt) \wedge d} \right] \right) \\
- \left( \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right]_t - (1 - F)^{dt} \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right]_{t - dt} \right) \right\}
\]
Higher central moments of \( \ln(r_t) \) are even messier, but tractable in principle. Asymptotically, things simplify a little.
Higher central moments of $\ln(r_t)$ are even messier, but tractable in principle. Asymptotically, things simplify a little.

Each $\mu_{2j}$ requires evaluation of terms such as $\mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^m \right]$.
Higher Moments

- Higher central moments of $\ln(r_t)$ are even messier, but tractable in principle. Asymptotically, things simplify a little.

- Each $\mu_{2j}$ requires evaluation of terms such as $\mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^m \right]$

- These in turn can be analyzed into terms of the form

$$\mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^{2n} \prod_{k=1}^{n} \left( \mathbb{E} \left[ e_j^{2k} \right] \right)^{nk} \right]$$

where $\sum_{k=1}^{n} kn_k \leq m - n$. 


Higher Moments

- Higher central moments of \( \ln(r_t) \) are even messier, but tractable in principle. Asymptotically, things simplify a little.

- Each \( \mu_{2j} \) requires evaluation of terms such as \( \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^m \right] \)

- These in turn can be analyzed into terms of the form
  \[
  \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^{2n} \prod_{k=1}^{n} \left( \mathbb{E} \left[ e_j^{2k} \right] \right)^{n_k} \right], \text{ where } \sum_{k=1}^{n} kn_k \leq m - n.
  \]

- Let \( \nu(x) = x^{2n} \prod_{k=1}^{n} \left( \mathbb{E} \left[ x^{2k} \right] \right)^{n_k} \) for any \( x \). As \( t \to \infty \) \( \mathbb{E} \left[ \sum_{j=1}^{\infty} \nu(e_j) \right] \) turns out to equal

  \[
  \frac{\mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right] \mathbb{E} \left[ \nu \left( 1 - (1 - F)^d \right) \right]}{1 - \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right]} + \mathbb{E} \left[ \nu \left( 1 - (1 - F)^d \right) \right]
  \]
Here \( d \) follows our \( \gamma(\alpha, \beta) \) interarrival distribution and \( \bar{d} \) (same as before) is its equilibrium distribution. Those expected values are all Laplace transforms that we know how to calculate.
Higher Moments

Here $d$ follows our $\text{gamma}(\alpha, \beta)$ interarrival distribution and $\mathbf{d}$ (same as before) is its equilibrium distribution. Those expected values are all Laplace transforms that we know how to calculate.

Using these, and the Edgeworth-based approximation, we put together formulae for the skewness, kurtosis, and 6-thosis of the modeled interest rate $r_t$. 

Looking at results, it seems more reasonable to use parameters that put the historical kurtosis within a sampling error, rather than forcing equality.
Higher Moments

- Here \( \mathbf{d} \) follows our \textit{gamma}(\( \alpha \), \( \beta \)) interarrival distribution and \( \overline{\mathbf{d}} \) (same as before) is its equilibrium distribution. Those expected values are all Laplace transforms that we know how to calculate.

- Using these, and the Edgeworth-based approximation, we put together formulae for the skewness, kurtosis, and 6-thosis of the modeled interest rate \( r_t \).

- Then we played games with EXCEL Solver to find parameters \( F, \sigma, \alpha, \) and \( \beta \) to reproduce asymptotically the historical variance, and kurtosis of \( r_t \) as well as the historical volatility (the standard deviation of \( \ln(r_t) - \ln(r_{t-\Delta t}) \)). We set \( \mu_T \) so that \( \mathbb{E}[T_j] \) equals the historical mean of \( r_t \) which, together with the drift compensation, assured that asymptotically the model would reproduce the historical mean of \( r_t \).
Here \( \textbf{d} \) follows our \textit{gamma}(\( \alpha, \beta \)) interarrival distribution and \( \overline{\textbf{d}} \) (same as before) is its equilibrium distribution. Those expected values are all Laplace transforms that we know how to calculate.

Using these, and the Edgeworth-based approximation, we put together formulae for the skewness, kurtosis, and 6-thosis of the modeled interest rate \( r_t \).

Then we played games with EXCEL Solver to find parameters \( F, \sigma_T, \alpha, \) and \( \beta \) to reproduce asymptotically the historical variance, and kurtosis of \( r_t \) as well as the historical volatility (the standard deviation of \( \ln(r_t) - \ln(r_{t-\Delta t}) \)). We set \( \mu_T \) so that \( \mathbb{E}[T_j] \) equals the historical mean of \( r_t \) which, together with the drift compensation, assured that asymptotically the model would reproduce the historical mean of \( r_t \).

Looking at results, it seems more reasonable to use parameters that put the historical kurtosis within a sampling error, rather than forcing equality.
With No Regime Switching, Just Match Historical Mean and Variance: $F = 0.0528$ (N=662)
With No Regime Switching, Just Match Historical Mean and Variance: $F = 0.0528 \ (N=662)$
With No Regime Switching, Just Match Historical Mean and Variance: $F=0.0528 \ (N=662)$
With No Regime Switching, Just Match Historical Mean and Variance: $F=0.0528$ (N=662)
With No Regime Switching, Just Match Historical Mean and Variance: $F = 0.0528$ (N=662)
With No Regime Switching, Just Match Historical Mean and Variance: $F = .0528$ (N=4,500)
With No Regime Switching, Just Match Historical Mean and Variance: $F = 0.0528 \ (N = 4,500)$
With No Regime Switching, Just Match Historical Mean and Variance: $F = 0.0528$ (N=4,500)
Match historical kurtosis:

$$F = 0.3993 / \alpha = 3 / \beta = 0.5 / \sigma_T = 0.6512 \text{ (N=662)}$$
Match historical kurtosis:

\[ F = \frac{.3993}{\alpha} = \frac{3}{\beta} = \frac{.5}{\sigma_T} = .6512 \ (N=662) \]
Match historical kurtosis:

\[ F = \frac{.3993}{\alpha} = 3 \quad \beta = \frac{.5}{\sigma_T} = .6512 \quad (N=662) \]
Match historical kurtosis:

\[ F = \frac{0.3993}{\alpha} = 3 / \beta = 0.5 / \sigma_T = 0.6512 \text{ (N=662)} \]
Match historical kurtosis: 

\[ F = \frac{.3993}{\alpha} = \frac{3}{\beta} = \frac{.5}{\sigma_T} = .6512 \text{ (N=5,000)} \]
Match historical kurtosis:

\[ F = 0.3993/\alpha = 3/\beta = 0.5/\sigma_T = 0.6512 \text{ (N=5,000)} \]
Kurtosis 74\% of Lognormal (vs historical 58\%)

\[ F = 0.1813/\alpha = 2.5/\beta = 1.25/\sigma_T = 0.6095 \ (N=662) \]
Kurtosis 74% of Lognormal (vs historical 58%)

\[ F = \frac{.1813}{\alpha} = \frac{2.5}{\beta} = \frac{1.25}{\sigma_T} = .6095 \ (N=662) \]
Kurtosis 74% of Lognormal (vs historical 58%)

\[ F = 0.1813/\alpha = 2.5/\beta = 1.25/\sigma_T = 0.6095 \quad (N=662) \]
Kurtosis 74% of Lognormal (vs historical 58%)

\[ F = 0.1813/\alpha = 2.5/\beta = 1.25/\sigma_T = 0.6095 \ (N=5,000) \]
Kurtosis 74% of Lognormal (vs historical 58%)

\[ F = 0.1813/\alpha = 2.5/\beta = 1.25/\sigma_T = 0.6095 \ (N=5,000) \]
Kurtosis 74% of Lognormal (vs historical 58%)

\[ F = 0.1813 / \alpha = 2.5 / \beta = 1.25 / \sigma_T = 0.6095 \]
Kurtosis 74% of Lognormal (vs historical 58%)

\[ F = 0.1813 \]
\[ \alpha = 2.5 \]
\[ \beta = 1.25 \]
\[ \sigma_T = 0.6095 \]
Kurtosis 90% of Lognormal (vs historical 58%)

\[ F = \frac{.1136}{\alpha} = \frac{2}{\beta} = \frac{2.07}{\sigma_T} = .5587 \quad (N=662) \]

Mean and Std Dev of \( r(t) \)
Kurtosis 90% of Lognormal (vs historical 58%)

\[ F = 0.1136 / \alpha = 2 / \beta = 2.07 / \sigma_T = 0.5587 \ (N=662) \]
Kurtosis 90% of Lognormal (vs historical 58%)

\[ F = 0.1136 / \alpha = 2 / \beta = 2.07 / \sigma_T = 0.5587 \ (N=662) \]
Kurtosis 90% of Lognormal (vs historical 58%)

\[ F = \frac{.1136}{\alpha} = \frac{2}{\beta} = \frac{2.07}{\sigma_T} = .5587 \quad (N=5,000) \]
Kurtosis 90% of Lognormal (vs historical 58%)

\[ F = \frac{.1136}{\alpha} = 2 / \beta = 2.07 / \sigma_T = .5587 \ (N=5,000) \]
Kurtosis 90% of Lognormal (vs historical 58%)

$$F = \frac{.1136}{\alpha} = 2 / \beta = 2.07 / \sigma_T = .5587$$
Kurtosis 90% of Lognormal (vs historical 58%)

\[ F = 0.1136 / \alpha = 2 / \beta = 2.07 / \sigma_T = 0.5587 \]