(1) State and prove the central limit theorem for i.i.d. random variables (any proof that does not use another CLT is acceptable).

(2) State and prove the Borel-Cantelli lemma.

(3) Prove that a.s. convergence implies convergence in probability.

(4) (a) State the Three Series Theorem.

   (b) Let $X_i$ be i.i.d. standard Poisson random variables. Find and prove the necessary and sufficient conditions for sequences of positive numbers $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ so that the series $\sum_{n=1}^\infty a_n X_n - b_n$ converges a.s.

   (c) Under which necessary and sufficient conditions is the sequence $Y_m = \sum_{n=1}^m a_n X_n - b_n$ a convergent supermartingale?

(5) Let $X$ be a non-negative random variable with $\mathbb{E}X^p < \infty$ for some $p > 0$, and $\mathcal{F}_n$ be an increasing sequence of $\sigma$-fields.

   (a) Prove that $\mathbb{E}(\sup_n \mathbb{E}\{X|\mathcal{F}_n\})^r < \infty$ if $p \geq 1$ and $0 < r < p$.

   (b) Prove the same if $r = p > 1$. 