Methods of Integration

Solutions to these problems should show all of your work, not just a single final answer.

Part 1: Integration by parts. Do each problem as follows: (1) specify \( u \) and \( dv \), (2) compute \( du \) and \( v \), (3) use integration by parts with your choice of \( u \) and \( dv \). (4) If you need integration by parts more than once, each time go through steps 1, 2, and 3 again.

Example. Compute \( \int x^2 e^x \, dx \).

Solution.

(1) Set \( u = x^2 \) and \( dv = e^x \, dx \).

(2) We have \( du = 2x \, dx \) and \( v = e^x \).

(3) Now \( \int x^2 e^x \, dx = \int u \, dv = uv - \int v \, du = x^2 e^x - \int e^x (2x) \, dx = x^2 e^x - 2 \int xe^x \, dx \).

(4) To find \( \int xe^x \, dx \), set \( u = x \) and \( dv = e^x \, dx \), so \( du = dx \) and \( v = e^x \). Then \( \int xe^x \, dx = \int u \, dv = uv - \int v \, du = xe^x - \int e^x \, dx = xe^x - e^x \).

(5) Substituting the result of (4) into (3),

\[
\int x^2 e^x \, dx = x^2 e^x - 2(\int xe^x \, dx) + C = (x^2 - 2x + 2)e^x + C.
\]

1. Compute \( \int x \cos(5x) \, dx \).
2. Compute $\int x^2 2^x \, dx$. (Hint: You can find an antiderivative of $2^x$ by recalling how to differentiate $2^x$.)
Part 2: Integration of rational functions.

Example. Compute \( \int \frac{2x + 1}{x^2 - 4} \, dx \) using partial fractions.

Solution. Write \( \frac{2x + 1}{x^2 - 4} = \frac{A}{x + 2} + \frac{B}{x - 2} \) for some \( A \) and \( B \). Clearing the denominator, \( 2x + 1 = A(x - 2) + B(x + 2) \). Setting \( x = 2 \) we get \( 5 = 4B \), so \( B = \frac{5}{4} \). Setting \( x = -2 \) we get \( -3 = -4A \), so \( A = \frac{3}{4} \). Thus \( \frac{2x + 1}{x^2 - 4} = \frac{3/4}{x + 2} + \frac{5/4}{x - 2} \), so

\[
\int \frac{2x + 1}{x^2 - 4} \, dx = \int \left( \frac{3/4}{x + 2} + \frac{5/4}{x - 2} \right) \, dx = \frac{3}{5} \ln |x + 2| + \frac{5}{4} \ln |x - 2| + C.
\]

3. Compute \( \int \frac{10}{x^3 - x^2 - 6x} \, dx \) using partial fractions.

4. Compute \( \int \frac{x^2 + x + 1}{x(x^2 + 4)} \, dx \) using partial fractions.
Part 3: Approximate Integration.

Example. (a) Compute the trapezoid approximation to \( \int_1^3 \sqrt{x} \, dx \) using \( n = 4 \) subintervals, rounding your approximation to 5 digits after the decimal point.

(b) Use the error bound for the trapezoid rule to determine an \( n \) such that the trapezoid approximation is guaranteed by the error bound to be within .01 of the value of the integral.

Solution.
(a) The trapezoid approximation with \( n = 4 \) is
\[
\frac{b - a}{2n} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) = \frac{2}{8} (f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3))
\approx 2.79306.
\]

(b) An upper bound on the error from the trapezoid rule with \( n \) intervals is \( \frac{K(b-a)(\Delta x)^2}{12} \), where \( \Delta x = (b-a)/n \) and \( K \) is an upper bound on \( |f''(x)| \) for all \( x \) in \([a,b]\). In our problem, \( f(x) = \sqrt{x} \), so \( f''(x) = -\frac{1}{4}x^{-3/2} \). For \( 1 \leq x \leq 3 \), we have \( x^{-3/2} \leq 1 \), so \( |f''(x)| \leq 1/4 \) when \( 1 \leq x \leq 3 \). Thus we can use \( K = 1/4 \), so the trapezoid error bound is \( \frac{(1/4)(3^{-1/2})^2}{12n^2} = \frac{1}{6n^2} \). Having the error be less than .01 means
\[
\frac{1}{6n^2} < .01 \iff n^2 > \frac{1}{6(.01)} \iff n > \sqrt{\frac{1}{.06}} \approx 4.082,
\]
so for \( n \geq 5 \) the trapezoid approximation will be within .01 of the integral.

5. (a) Compute the trapezoid approximation to \( \int_2^3 x \sin x \, dx \) using \( n = 4 \) subintervals, rounding your approximation to 5 digits after the decimal point. (Remember to set your calculator to radian mode for trigonometric functions.)

(b) Use the error bound for the trapezoid rule to determine an \( n \) such that the trapezoid approximation is guaranteed by the error bound to be within .01 of the value of the integral.
6. (a) Compute the Simpson’s rule approximation to \( \int_{1}^{2} \sqrt{x} \, dx \) using \( n = 4 \) subintervals, rounding your approximation to 5 digits after the decimal point.

(b) Use the error bound for Simpson’s rule to determine an \( n \) such that the Simpson’s rule approximation is guaranteed by the error bound to be within \( 10^{-6} \) of the value of the integral. (Remember \( n \) must be even.)
Part 4: Improper Integrals.

7. For $a > 0$, compute the improper integrals $\int_0^{\infty} e^{-ax} \, dx$ and $\int_0^{\infty} xe^{-ax} \, dx$. Your answer will be in terms of $a$.

8. Compute the improper integral $\int_0^{\infty} \frac{dx}{(x + 2)(x + 5)}$ using partial fractions.
9. Decide if the improper integral \( \int_0^\infty \frac{x}{x^2 + 1} \, dx \) is convergent or divergent. If it is convergent, evaluate it.
Optional Question.

10. Vibrations show up in many places: civil engineering (oscillations in a bridge or the reaction of a building to an earthquake), music (sound is a vibration of pressure waves), and ski design (smaller vibrations make a smoother ride). The following computation is fundamental in any mathematical study of vibrations: for all positive integers $m$ and $n$, use integration by parts to show

$$
\int_{0}^{2\pi} \sin(mx)\cos(nx)\,dx = 0.
$$

(Hint: Use the bounds of integration during the integration by parts, and treat $m = n$ and $m \neq n$ separately. It may help to first try this for specific $m$ and $n$, such as $m = 2$ and $n = 3$, and then $m = 5$ and $n = 5$.) There are two other integral formulas related the one above, with products of two sines and two cosines:

$$
\int_{0}^{2\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 
\pi, & \text{if } m = n, \\
0, & \text{if } m \neq n
\end{cases}
$$

$$
\int_{0}^{2\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 
\pi, & \text{if } m = n, \\
0, & \text{if } m \neq n
\end{cases}
$$

Here too $m$ and $n$ are positive integers. The optional question is only the first formula.