Parametric Curves

Part 1: Parametrizations.

1. In each part of this problem,

(i) eliminate the parameter to obtain an equation in terms of $x$ and $y$,

(ii) graph the parametric curve over the indicated range of $t$-values, including arrows on the curves to indicate the direction of increasing values of $t$ and labeling the points on the curve where $t$ has its first value, its last value, and its middle value. The middle $t$-value need not correspond to the middle position along the curve.

(a) $x = \sqrt{t} + 4, \ y = 3\sqrt{t}$ for $0 \leq t \leq 16$.

(b) $x = 3\cos t, \ y = 3\sin t$ for $0 \leq t \leq \frac{\pi}{2}$. 
2. Write down a parameterization of the following circles and then draw an arrow on the graph indicating how the parametrization traces out the circle. Mark the starting point as $t = 0$.

(a) A circle centered at the origin with radius 16, traversed counterclockwise starting at $(16, 0)$.

(b) A circle centered at $(2, 3)$ with radius 2, traversed clockwise starting at $(2, 5)$. 
3. An ellipse centered at the origin has major radius 5 along the \(x\)-axis and minor radius 2 along the \(y\)-axis.

(i) Give a parameterization of the ellipse so that \(0 \leq t \leq 10\) traces out the ellipse once counterclockwise starting at \((5, 0)\), draw a graph with arrows marking the direction of increasing \(t\) on the ellipse, and mark the point where \(t = 7\).

(ii) Use calculus and the parametrization you found in (i) to compute the slope of the tangent line to the ellipse at the point where \(t = 7\), both \textit{exactly} in terms of values of trigonometric functions and as an approximation to two decimal places.
Part 2: Polar Coordinates.

4. In each part below, $P$ is a point in polar coordinates. Do the following:

   (i) Convert $P$ to Cartesian coordinates (give exact values, not approximations).
   (ii) Plot $P$ in the $xy$-plane.
   (iii) Give two additional representations of $P$ in polar coordinates.

(a) $P = \left( 2, \frac{\pi}{4} \right)$

(b) $P = \left( -3, \frac{3\pi}{4} \right)$

(c) $\left( -1, -\frac{\pi}{3} \right)$
5. (a) On separate axes plot the two polar equations $r = 1 + 2 \sin \theta$ and $r = 1 - 2 \sin \theta$, drawing arrows on each curve to indicate the direction traced out as $\theta$ increases.

(b) Use calculus and polar coordinates to find the equation of the tangent line to $r = 1 + 2 \sin \theta$ at the point $(x, y) = (1, 0)$, which you graphed in the previous part. (Hint: On this curve write $x$ and $y$ in terms of $\theta$.)
6. Below is the 3-leaf rose \( r = 2 \cos(3\theta) \).

\begin{center}
\begin{tikzpicture}
\begin{scope}[scale=0.5,rotate=90]
\draw[->] (-3.5,0) -- (3.5,0) node[anchor=north] {\(x\)};
\draw[->] (0,-3.5) -- (0,3.5) node[anchor=east] {\(y\)}; 
\draw (0,0) circle (3cm);
\end{scope}
\end{tikzpicture}
\end{center}

(a) Mark points on the graph where \( \theta = 0 \) and \( \theta = \pi/3 \), and draw arrows on the rose to indicate the direction in which it is traced out as \( \theta \) increases.

(b) Determine the smallest positive angle \( \theta \) at which the rose passes through the origin, and use this to help you set up an integral for the area of the right leaf.

(c) Compute the integral in part b. (Hint: \( \cos(2\alpha) = 2\cos^2\alpha - 1 \), so \( \cos^2\alpha \) can be written in terms of \( \cos(2\alpha) \), which is easier to integrate.)
7. Below is the graph of the limaçon $r = 1 + 2 \cos \theta$.

(a) Mark the points on the curve where $\theta$ is $0$, $\pi/2$, $\pi$, $3\pi/2$, and $2\pi$ and use this to draw arrows on the curve indicating the direction in which it is traced out as $\theta$ increases.

(b) Determine all $\theta$ between 0 and $2\pi$ where the curve passes through the origin.

(c) Set up, but do not evaluate, an integral for the area of the inner loop of the limaçon entirely in terms of $\theta$. Be sure the bounds of integration are right.