Section 3.10: Related Rates

Example 1: A 6 ft. man is walking away from a 20 ft. tall street light. If the man is walking at a rate of 4 ft/sec how fast will the length of his shadow be changing when he is 30 ft. from the light.

Let $x$ be the distance between the man and the street light and let $s$ be the length of his shadow. GIVEN: $\frac{dx}{dt} = 4 \text{ ft/sec}$. WANT: $\frac{ds}{dt}$ when $x = 30$.

Note that we have two similar triangles: the overall triangle with height 20 and base length $s + x$ and the smaller one inside of it with height 6 and base length $s$. We want to solve for $s$ in terms of $x$ by using the ratio

$$\frac{6}{s} = \frac{20}{s + x}$$

$6(s + x) = 6s + 6x = 20x$

$6s = 14x$

$s = \frac{3}{7}x$

Differentiate both sides to get

$$\frac{ds}{dt} = 3 \frac{dx}{dt} = \frac{3}{7}(4) = \frac{12}{7}$$

So it turns out that as long as the man is moving at a constant rate, the rate of his shadow will also remain constant $\frac{12}{7}$ ft/sec. regardless of how far away he is from the street light.
Example 2: A cone-shaped funnel is emptying water at a rate of $5\text{cm}^3/\text{hr}$. The base (positioned at the top) radius is 20 cm and the height is 10 cm. At what rate is the depth of the water in the funnel changing when the depth of the water is 6 cm?

Let $V$ be the volume of water left in the funnel, $r$ and $h$ the radius and height of the water at any given point in time. GIVEN: $\frac{dV}{dt} = 5 \text{ cm}^3/\text{hr}$. WANT: $\frac{dh}{dt}$ when $h = 6 \text{ cm}$.

The general equation for the volume of a cone is

$$V = \frac{1}{3} \pi r^2 h$$

We want to rewrite this only in terms of $V$ and $h$ since those are the variables we are given information about. As in the first example, we have similar triangles, one with height 10 and width 20, the smaller with height $h$ and width $r$. Thus,

$$\frac{20}{10} = \frac{r}{h}$$

$$20h = 10r$$

$$2h = r$$

Plug this back into the equation for $V$ so that it’s just in terms of $h$

$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi (2h)^2 h = \frac{4}{3} \pi h^3$$

Differentiate with respect to time, solve for $\frac{dh}{dt}$, and plug in values:

$$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{4\pi h^2} \frac{dV}{dt} = \frac{1}{4\pi (6\text{ cm})^2} (5\text{ cm}^3/\text{hr}) \approx 0.0111 \text{ cm/ hr}$$
Example 3: An observer stands 300 ft from a hot-air balloon that is rising upwards at 20 ft/sec. What rate is the angle between the observer’s line of sight of the balloon and the ground changing when the balloon is 400 ft above the ground?

Let \( y \) be the vertical distance that the balloon has risen, \( \theta \) the angle created between the line of sight of the observer and the ground. GIVEN: \( \frac{dy}{dt} = 20 \text{ ft/sec} \). WANT: \( \frac{d\theta}{dt} \) when \( y = 400 \) ft.

\[
\tan(\theta) = \frac{y}{300} \Rightarrow \theta = \arctan\left(\frac{y}{300}\right)
\]

Recall: \( \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \)

\[
\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{y}{300}\right)^2} \left( \frac{1}{300} \frac{dy}{dt} \right) = \frac{1}{300 + \frac{y^2}{300}} \frac{dy}{dt}
\]

So, when \( y = 400 \),

\[
\frac{d\theta}{dt} = \frac{1}{300 + \frac{400^2}{300}} (20) \approx 0.024 \text{ radians/sec}.
\]
Example 4: An 8 ft tall projector is positioned 2 feet above eye level. If you are walking towards the screen at a rate of 4 ft/sec, at what rate will your viewing angle be changing when you are 20 ft away?

Define \( x \) to be the distance between the observer and the projector and \( \theta \) the angle of observation. GIVEN: \( \frac{dx}{dt} = -4 \) ft/s. WANT: \( \frac{d\theta}{dt} \) when \( x = 20 \).

Note that we have two similar triangles, one with base \( x \) and height 10, the other with base \( x \) and height 2. They share the angle \( \theta_b = \theta + \theta_s \).

\[
\theta = \theta_b - \theta_s
\]

\[
\theta = \arctan\left(\frac{10}{x}\right) - \arctan\left(\frac{2}{x}\right)
\]

\[
\frac{d\theta}{dt} = \left(\frac{1}{1 + \left(\frac{10}{x}\right)^2}\right)\frac{-10}{x^2}\frac{dx}{dt} - \left(\frac{1}{1 + \left(\frac{2}{x}\right)^2}\right)\frac{-2}{x^2}\frac{dx}{dt}
\]

\[
\Rightarrow \frac{d\theta}{dt} = \frac{-10}{x^2 + 100}\frac{dx}{dt} + \frac{2}{x^2 + 4}\frac{dx}{dt}
\]

So, when \( x = 20 \),

\[
\frac{d\theta}{dt} = \frac{-10}{(20)^2 + 100}(-4) + \frac{2}{(20)^2 + 4}(-4) \approx 0.0602\text{ radians/sec.}
\]