Mathematics in the Arab and Persian World in the Middle Ages

There were many scholars in the Moslem world who worked on mathematics or sciences which use mathematics, as well as many who translated ancient classics from Greek and Sanskrit into Arabic and thus preserved the knowledge of long ago, much to the benefit of thinkers of later eras. Under some national rulers such as caliphs Harun al-Rashid and al-Ma’mun, scholarship was able to flourish. And in such good times, intellectuals were encouraged to study and interact in the Houses of Wisdom set up for the purpose. Here we summarize facts about some of the famous or outstanding mathematical writers of those times. Some of them were born in Arabia, many in Persia, some in Turkey, and others in central Asia. For more details, see Chapter 7 of A History of Mathematics: An Introduction by Victor Katz.

Mohammed ibn Musa al-Khowarizmi (Mohammed, son of Moses of Khwarezm) wrote two very influential books. One was about the Hindu numerals, the other (written about 830) about algebra. The latter was called The Science of Restoration and Opposition or Completion and Balancing; its brief name in Arabic was Hisab al-jabr w’al-muqabalah, from which is derived the word “algebra.” In medieval times, algebristas were men who reset people’s broken bones. al-Khowarizmi’s name also led to the word “algorithm.”

Abu-l-Hasan Thabit ibn Qorra (Thabit, father of Hassan, son of Qorra) (826-901) declared that in order to be valid, algebra must be validated by geometric proofs based on the work of Euclid. Thabit also discovered a method for searching for amicable numbers, which we will examine shortly.

Abu-l-Wefaa (940-998) translated the writings of Diophantus. The Indian astronomer-mathematicians had invented the sine and cosine functions. Abu-l-Wefaa introduced the tangent function into trigonometry. He also made tables of values of the sine and tangent for intervals of 15' = 1/4 of a degree.

Abû Kâmil (10th century) wrote an advanced book on algebra which was a source of ideas for Fibonacci in the early 1200s.

al-Karkhi (11th century) wrote the Fakhrî, which he dedicated to his patron, Fakhir al-Mulk, the vizier of Baghdad. This is an advanced book on algebra. al-Karkhi looked for rational roots of the three variable equation \( x^3 + y^3 = z^2 \) and and expressed the components of the solutions as rational functions of two variables which range over the rational numbers (what he really did was to take the ‘surface’ in three dimensional Cartesian space given by this equation and find parametric equations for it): \( x = \frac{u^2}{1 + v^3}, \quad y = \frac{u^2v}{1 + v^3}, \quad z = \frac{u^3}{1 + v^3}. \) (Note the singularity at \( v = -1. \)) It is very unusual for a global surface to admit a parametrization by rational functions, so this was quite an accomplishment.

Omar Khayyam (1048-1123) was a poet as well as a mathematician. See the attached notes.

Nasir ed-din (= Abu Jafar Muhammad ibn al-Hassan Nasir al-Din al-Tusi), scientist (1201-1274), studied Euclid’s parallel postulate; his discussion influenced G. Saccheri (18th century), who came close to discovering non-Euclidean geometry. He also wrote on plane and spherical geometry. (There were several Persian scholars who were born in the region of Tus in Khorasan (northeastern Iran), including Sharaf al-Tusi (1135-1213).)

Ulugh Beg and Jamshid al-Kashi both lived in Samarkand in the fifteenth century. For his astronomical calculations, Ulugh Beg improved the tables of Abu-l-Wefaa, finding sines and tangents for intervals of \( 1' = \frac{1}{60} \) of a degree, accurate to 8 or more decimal places, while al-Kashi found \( \pi \) correct to 16 decimal places (around the year 1425).
Thābit ibn Qorra and Amicable Number Pairs

Thābit was famous in his time as a linguist, mathematician, philosopher, and physician. He translated the *Elements* and works of Apollonius, Archimedes, Caludius Ptolemy, and others from Greek into Arabic and wrote about astronomy, magic squares, and amicable numbers. Let’s quote from Oystein Ore’s book, *Number Theory and Its History*: “In Arab mathematical writings the amicable numbers occur repeatedly. They play a role in magic and astrology, in the casting of horoscopes, in sorcery, in the concoction of love potions, and in the making of talismans.” Ore then quotes Ibn Khaldun, the famous 14th century Arab scholar, who wrote about the magical properties of 220 and 284, “the amicable or sympathetic numbers.”

Recall that whole numbers $m$ and $n$ are amicable if and only if $\sigma(m) = \sigma(n) = m + n$, where $\sigma$ is the sum of the positive divisors of a number. (I.e., each is the sum of the proper divisors of the other.) Ore says “In ancient numerology there appears but single set of amicable numbers, namely the pair $N = 220 = 2^2 \cdot 5 \cdot 11, M = 284 = 2^2 \cdot 71$.”

Thābit discovered a rule for searching for pairs of amicable numbers, which goes as follows. Make a table with the whole numbers in the first row and below each integer $n$, write the number $p_n = 3 \cdot 2^n - 1$. (An important observation made by Ore is “As may be seen, each number is obtained by doubling the preceding and adding 1.” This simplifies the task of generating the table.) Thābit’s rule is now this: If for some index $n$, two consecutive terms $p_{n-1}$ and $p_n$ are prime numbers, then examine the number $q_n = 9 \cdot 2^{2n-1} - 1$. If $q_n$ is also prime, then the pair $N = 2^n p_{n-1} p_n$ and $M = 2^n q_n$ is amicable.

The proof of this statement in Ore’s book uses the multiplicative property of the sum of divisors function sigma, as follows: $\sigma(N) = \sigma(2^n) \sigma(p_{n-1}) \sigma(p_n)$ and $\sigma(M) = \sigma(2^n) \sigma(q_n)$, followed by an involved calculation.

It is strange but true that we don’t have evidence that Thābit actually applied his rule and found any amicable pairs other than the known pair 220 and 284, which correspond to $n = 2$ and the primes $p_1 = 5, p_2 = 11, q_2 = 71$. The next consecutive $p$’s are $p_2 = 11, p_3 = 23$, both of which are prime; but $q_3 = 284$ is not prime, so the rule doesn’t apply. In 1636, Pierre de Fermat rediscovered Thābit’s rule and used it to find another amicable pair, 17296 and 18416, generated by $p_3 = 23, p_4 = 47, q_4 = 1151$, all of which are primes. Marin Mersenne published Fermat’s discovery in his book *Seconde Partie de l’Harmonie Universelle*. (In letters he wrote to Mersenne two years later, René Descartes stated that he had found the same rule and had used it to find another amicable pair: 9,363,584 and 9,437,056. These are given by Thābit’s rule with $n = 7$.)

In the 18th century, Leonhard Euler “took up the search for amicable numbers in a systematic manner and developed several methods for finding them. In 1747 he gave a list of 30 pairs which he later expanded to more than 60.” (Thābit’s rule doesn’t produce all possible amicable numbers. In fact, in 1866 the teenager Nicolo Paganini, who came later than the famous violinist and composer of the same name, reported that 1184 and 1210 are amicable.) Euler was also the first person to find a pair of odd amicable numbers, 69615 and 87633. In 1988 the first examples were given of 15 pairs of (very large) odd amicable numbers that are not multiples of 3. The authors of that article asked an interesting question: Is there a pair of amicable numbers with one odd and the other even? That seems to be a very hard problem.
Omar Khayyam

Arab scholars of the Middle Ages are known primarily for their role in preserving ancient knowledge. “By the 10th century nearly all the texts of Greek science that were to become known to the Western world were available in Arabic.” “The new science which began to percolate into Western Christendom in the 12th century was largely Arabic in form, but it was founded on the works of the ancient Greeks.” (A.C. Crombie, Medieval and Early Modern Science, v.1, Ch. II.)

But Arab and Persian thinkers also made original contributions of their own. Among these was the Persian poet and philosopher Ghiyathuddin Abu’lfath ‘Omar ibn Ibrahim al-Khayyami, or Omar Khayyam. Supported by a yearly salary, essentially a research grant, Omar pursued mathematical and scientific studies and wrote a text on algebra which became a standard in Arabic. In 1074-1079 he worked on the revision of the calendar and also provided the necessary astronomical observations.

Omar Khayyam also wrote about 1000 ruba’is, or quatrains (four-line stanzas), each composed on a particular occasion and a self-contained poem in itself. These show his love of wine and women and philosophy as well as his religious skepticism. A beautiful, sometimes faithful, translation of 97 of the quatrains by Edward Fitzgerald (1859), The Rubaiyat of Omar Khayyam, made his work well known in the English speaking world. “A book of verses underneath the bough, a loaf of bread, a jug of wine and thou, beside me dreaming in the wilderness, etc.” – the words were known to millions who would have had no idea of how to intersect a parabola and a hyperbola, as Omar did to solve certain kinds of cubic equations.

In his book Risala, issued circa 1079, Omar Khayyam dealt with polynomial equations and analyzed them both algebraically and geometrically. Unlike the Indian mathematicians, he used no symbols for variables (or any other mathematical terms) but wrote out everything in words. Since he had no concept of zero or negative numbers, he needed to study many forms of the equations and accepted only the positive roots. Often, he was able to solve all equations of a certain form, but he did solve one specific quartic equation: \((100 - x^2)(10 - x)^2 = 8100\), by intersecting a circle and a hyperbola. Some kinds of cubic equations he solved are \(x^3 + b^2x = b^2c\) – by intersecting a parabola and a semi-circle (notice the balancing of volume units on both sides of the equation); \(x^3 + ax^2 = c^3\) – by intersecting a parabola and a hyperbola; and \(x^3 + ax^2 + b^2x = b^2c\) – by intersecting an ellipse and a hyperbola. Obviously, he had a good background in conic sections, probably from reading Apollonius’ work in Arabic. Omar had theoretical insights into what was or was not possible in algebra: (a) he showed that a cubic equation could have no roots, only one root, two roots, or three roots; (b) he stated a belief that the general cubic equation could not be solved algebraically nor the general quartic equation be solved geometrically. Although he was proved to be wrong some 420 years after his passing in 1123 (or perhaps December 1122), the idea that certain classes of equations could not be solved by formulas eventually bore fruit in the discoveries of Ruffini, Abel, and Galois in the first third of the 19th century .