Math 211 Exam 2
Nov 9, 2007

Name______________________(Please Print)PID______________________

1. A predator-prey system is modeled by

$$\begin{aligned}\frac{dR}{dt} &= 2R - 1.2RF \\ \frac{dF}{dt} &= -F + 0.9RF\end{aligned}$$

modify the system based on the following assumptions.

a. (8 pts) How could you modify the system to include the effect of hunting of predator at a rate proportional to the number of predators?

This assumption only affects the rate of change for the predator. The hunting of predator reduces the number of predators, we have $\frac{dR}{dt}$ stays the same yet $\frac{dF}{dt}$ decreases by a rate proportional to F .

$$\begin{aligned}\frac{dR}{dt} &= 2R - 1.2RF \\ \frac{dF}{dt} &= -F + 0.9RF - \alpha F.\end{aligned}$$

1. **b.** (8 pts) Suppose the predators find a food source that is limited in supply, how would you modify the system to include this effect?

A limited food source meaning without the prey, the predator do not decrease, it growth following logistic growth model. Therefore

$$\begin{aligned}\frac{dR}{dt} &= 2R - 1.2RF \\ \frac{dF}{dt} &= F\left(1 - \frac{F}{N}\right) + 0.9RF\end{aligned}$$

1. **c.** (8 pts) Suppose the prey move out of the area at a rate proportional to the number of predators in the area, how to modify the system?

This will affect the rate of change in R but doesn't affect the rate of change of F, therefore

$$\begin{aligned}\frac{dR}{dt} &= 2R - 1.2RF - \alpha F \\ \frac{dF}{dt} &= -F + 0.9RF.\end{aligned}$$

1. Consider the partially decoupled system

$$\begin{cases} \frac{dx}{dt} = x + 2y + 1 \\ \frac{dy}{dt} = 3y \end{cases}$$

a. (10 pts) Derive the general solution.

b. (6 pts) Find the equilibrium points of the system.

c. (6 pts) Find the solution that satisfies initial condition $(x_{0,}y_0) = (-1, 3)$.

Solution:
From the second equation, $y(t) = c_1 e^{3t}$, plug back into the first equation, we get

$$\frac{dx}{dt} = x + 2c_1 e^{3t} + 1$$

The integrating factor for this equation is $\mu(t) = e^{-\int 1dt} = e^{-t}$, multiply by e^{-t}, get

$$\left(e^{-t}x\right)' = 2c_1 e^{2t} + e^{-t}.$$

Integrating, we get

$$\begin{aligned} e^{-t}x(t) &= \int 2c_1 e^{2t} + e^{-t}dt \\ &= c_1 e^{2t} - e^{-t} + c_2 \end{aligned}$$

Therefore

$$x(t) = c_1 e^{3t} - 1 + c_2 e^t.$$

a. General solution is

$$\mathbf{Y}(t) = \begin{pmatrix} c_1 e^{3t} - 1 + c_2 e^t \\ c_1 e^{3t} \end{pmatrix}.$$

b. To find equilibrium, we set right hand side equal to zero.

$$\begin{aligned} x + 2y + 1 &= 0 \\ y &= 0 \end{aligned}$$

From the second equation, we get $y = 0$, plug into the first equation, we get $x = -1$.

c. $\mathbf{Y}(0) = \begin{pmatrix} c_1 - 1 + c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, from this we find $c_1 = 3$, $c_2 = -3$. The special solution is

$$\mathbf{Y}(t) = \begin{pmatrix} 3e^{3t} - 1 - 3e^t \\ 3e^{3t} \end{pmatrix}.$$

1.

2. True or False. Justify your answers. (6 pts each)

a. There is a real 2×2 matrix A that has a real eigenvalue and a complex eigenvalue.

b. $\mathbf{Y}(t) = (\cos 2t, \sin t)$ is a solution to some linear system.

c. A is a 2×2 matrix, then $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ can have three different straight line solutions.

d. $(1 + \cos 2t, \sin 2t)$ is a solution of

$$\begin{cases} \frac{dx}{dt} = -2y \\ \frac{dy}{dt} = 2x - 2 \end{cases}.$$

Then any solution with $(x_0, y_0) = \left(1, \frac{1}{2}\right)$ satisfies $0 < x(t) < 2$.

Solution :

a. False. Since complex eigenvalues come in pairs, and A have two eigenvalues (counting multiplicity). Therefore either two complex eigenvalues or two real eigenvalues, can not have one real and one complex.

b. False. Since $x(t)$ and $y(t)$ do not have same frequency, they can not be a solution to linear system. In fact,

$$\frac{dx}{dt} = -\sin 2t, \quad \frac{dy}{dt} = \cos t$$

They are not linearly related to x and y. Another way to see this is that A has two real eigenvalues or two complex eigenvalues. The only case where a solution oscillates is when A has two complex eigenvalues. However we know in this case $\lambda = \alpha \pm i\beta$, with the corresponding eigenvector is $\mathbf{V} = \mathbf{v}_1 + i\mathbf{v}_2$ for $\lambda = \alpha + i\beta$ and $\mathbf{V} = \mathbf{v}_1 - i\mathbf{v}_2$ for $\lambda = \alpha - i\beta$. Solution is

$$\mathbf{Y}(t) = e^{(\alpha + i\beta)t} (\mathbf{v}_1 + i\mathbf{v}_2).$$

We therefore know $x(t)$ and $y(t)$ should have same period.

c. True. For example, If $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ for $\lambda \neq 0$, the system $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ infinitely many straight line solutions. In fact, for each given initial data (x_0, y_0), the solution is $\mathbf{Y}(t) = \left(x_0 e^{\lambda t}, y_0 e^{\lambda t}\right)$, which is a straight line connecting $(0, 0)$ and (x_0, y_0).

d. True. Solution curve of $(1 + \cos 2t, \sin 2t)$ is a circle of radius 1 centered at (0,1). And $\left(1, \frac{1}{2}\right)$ lies inside this solution curve, by uniqueness theorem, solution with initial data $\left(1, \frac{1}{2}\right)$ must also lies inside the same solution curve, therefore satisfies $0 < x(t) < 2$.

1.

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \mathbf{Y}$$

a. (10 pts) Find its general solution.

b. (6 pts) Find the special solution with $\mathbf{Y}(0) = (1, -1)$

c. (8 pts) Sketch the graph of $x(t)$ and $y(t)$ for the special solution obtained in b.

d. (6 pts) Sketch the phase portrait of the system.

Solution:

Let $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5 = 0$, we get $(\lambda+1)^2 = -4$, hence $\lambda = -1 \pm 2i$. To find the corresponding eigenvector for $\lambda = -1 + 2i$, we solve

$$(A - \lambda I)\,\mathbf{v} = \begin{pmatrix} 1-2i & 1 \\ -5 & -1-2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

From this we have

$$y = (-1+2i)\,x$$

Pick $x = 1$, we get an eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1+2i \end{pmatrix}$$

Therefore a complex solution is

$$\begin{aligned} \mathbf{Y}(t) &= e^{(-1+2i)t} \begin{pmatrix} 1 \\ -1+2i \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} \cos 2t + i \sin 2t \\ (\cos 2t + i \sin 2t)(-1+2i) \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} \cos 2t + i \sin 2t \\ -\cos 2t - 2\sin 2t + i(2\cos 2t - \sin 2t) \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix} + i e^{-t} \begin{pmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{pmatrix} \end{aligned}$$

a. General solution is

$$\mathbf{Y}(t) = k_1 e^{-t} \begin{pmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{pmatrix}$$

b. Special solution with initial data $(1, -1)$

$$k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We can easily see that

$$k_1 = 1,\ k_2 = 0$$

Special solution is

$$\mathbf{Y}(t) = e^{-t}\begin{pmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix}.$$

c. The graph of $x(t)$ and $y(t)$ are periodically oscillating with period π and amplitude exponentially decreasing. See graph outside the answers of my office.

d. Since the vector field at initial date (1,-1) is

$$\begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix},$$

the phase portrait spirals clockwise in toward (0,0).