Math 211 Exam 1
Sep 28, 2007

Name____________________(Please Print)PID____________________

1. Consider the autonomous differential equation $dy/dt = f(y)$ where the graph of $f(y)$ is given below.

   a. (8 pts) Sketch the phase line for this equation and identify the equilibrium points as sinks, sources or nodes.

   b. (12 pts) Give rough sketches of the graph of the solutions that satisfy the initial conditions $y(0) = -3$, $y(0) = 0$, $y(0) = 2$.

Solution:

a. Since $f(-5) = f(-2) = f(1) = f(4) = 0$, $f(x) > 0$ for $x < -5$, $f(x) < 0$ for $-5 < x < -2$, $f(x) > 0$ for $-2 < x < 1$, $f(x) > 0$ for $1 < x < 4$ and $f(x) < 0$ for $x > 4$. The phase line is a vertical straight line with dots at 4,1,-2,-5. Arrow down to 4, up arrow between 1 and 4, up arrow between -2 and 1, down arrow between -2 and -5, up arrow below -5.

b. The rough sketch of solution with $y(0) = 2$ is an increasing curve bounded between $y = 4$ and $y = 1$. The rough sketch of solution with $y(0) = 0$ is an increasing curve bounded between $y = 1$ and $y = -2$. The rough sketch of solution with $y(0) = 2$ is a decreasing curve bounded between $y = -2$ and $y = -5$.

2 (20 pts) Consider the initial value problem $dy/dt = 3 - y^2$, $y(0) = 0$. Using Euler's method with $\Delta t = 0.5$, plot the graph of an approximate solution over the interval $0 \le t \le 2$.

Soluiton: $t_{k+1} = t_k + \Delta t$, $y_{k+1} = y_k + f(t_k, y_k)\Delta t = y_k + (3 - y_k^2)\Delta t$.

$$
\begin{aligned}
t_0 &= 0, \quad y_0 = 0 \\
t_1 &= 0.5, \quad y_1 = y_0 + \left(3 - y_0^2\right) 0.5 = 1.5 \\
t_2 &= 1, \quad y_2 = y_1 + \left(3 - y_1^2\right) 0.5 = 1.875 \\
t_3 &= 1.5, \quad y_3 = y_2 + \left(3 - y_2^2\right) 0.5 = 1.617125 \\
t_4 &= 2.0, \quad y_4 = y_3 + \left(3 - y_3^2\right) 0.5 \approx 1.81.
\end{aligned}
$$

Plot five points on the ty plane and connect five points by a curve.

3. Suppose a species of fish in a lake has population that is modeled by the logistic population model with growth rate k, carrying capacity N, and time t measured in years.

   a. (6 pts) Adjust the model if C fish are harvested each year.

   b. (8 pts) If $k = 0.3$, $N = 2400$, $C = -100$, find equlibrium solutions.

   c. (6 pts) If $k = 0.3$, $N = 2400$, $C = -100$ and $P(0) = 1500$, is it possible for the number of fish grow to 2200 at certain point?

   d. (8 pts) If $k = 0.3$, $N = 2400$, find value C where bifurcation appears.

Solution:

a.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - C$$

b.

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{2400}\right) - 100$$

Set

$$0.3P\left(1 - \frac{P}{2400}\right) - 100 = 0$$

This is a quadratic equation

$$-\frac{0.3}{2400}P^2 + 0.3P - 100 = 0$$

By quadratic formula,

$$\begin{aligned} P &= \frac{-0.3 \pm \sqrt{0.3^2 - 4 \times \left(-\frac{0.3}{2400}\right)(-100)}}{2 \times \left(-\frac{0.3}{2400}\right)} \\ &= \frac{0.3 \pm \sqrt{0.09 - 0.05}}{2 \times \frac{1}{8000}} \\ &= 400 \text{ or } 2000. \end{aligned}$$

c. No. Since $400 < P(0) < 2000$, $f(P) = 0.3P\left(1 - \frac{P}{2400}\right) - 100$ is continuous and has continuous derivative, by uniqueness theorem, we must have $400 < P(t) < 2000$ for all t. Therefore can never reach 2200.

d.

$$f_C(P) = 0.3P\left(1 - \frac{P}{2400}\right) - C$$

To find equilibrium solutions for a given C, we set

$$f_C(P) = 0.3P\left(1 - \frac{P}{2400}\right) - C = 0$$

by quadratic formula,

$$\begin{aligned}
P &= \frac{-0.3 \pm \sqrt{0.3^2 - 4 \times \left(-\frac{0.3}{2400}\right)(-C)}}{2 \times \left(-\frac{0.3}{2400}\right)} \\
&= \frac{0.3 \pm \sqrt{0.09 - \frac{C}{2000}}}{2 \times \frac{1}{8000}}.
\end{aligned}$$

Therefore for

$$\begin{aligned}
0.09 - \frac{C}{2000} &> 0, \frac{dP}{dt} = f_C(P) \text{ has two equilibrium solutions.} \\
0.09 - \frac{C}{2000} &= 0, \frac{dP}{dt} = f_C(P) \text{ has one equilibrium solution.} \\
0.09 - \frac{C}{2000} &< 0, \frac{dP}{dt} = f_C(P) \text{ has no equilibrium solution.}
\end{aligned}$$

From this, we see $0.09 - \frac{C}{2000} = 0$, i.e. $C = 180$ is a bifurcation value.

4. Consider differential equation $dy/dt = t^2 y + t^2$.

   a. (10 pts) Find its general solution by separating variables.

   b. (10 pts) Note this equation is also a nonhomogeneous linear equation. Find the general solution of its associated homogeneous equation $dy/dt = t^2 y$.

   c. (4 pts) Show that $y = -1$ is a solution of $dy/dt = t^2 y + t^2$.

   d. (8 pts) Using Extended Linearity Principle, find the general solution of the nonhomogeneous equation.

Solution:

a.

$$dy/dt = t^2(y+1)$$

i.e.

$$\frac{1}{y+1} dy/dt = t^2$$

integrating with respect to t, we get

$$\int \frac{1}{y+1} dy = \int t^2 dt.$$

$$\begin{aligned}
\ln|y+1| &= \frac{1}{3}t^3 + C \\
|y+1| &= e^{\frac{1}{3}t^3 + C} \\
y + 1 &= Ce^{\frac{1}{3}t^3} \\
y &= -1 + Ce^{\frac{1}{3}t^3}.
\end{aligned}$$

b.

$$dy/dt = t^2 y$$

i.e.

$$\frac{1}{y} dy/dt = t^2$$

integrating with respect to t, we get

$$\int \frac{1}{y} dy = \int t^2 dt.$$

$$\begin{aligned}
\ln |y| &= \frac{1}{3} t^3 + C \\
|y| &= e^{\frac{1}{3}t^3 + C} \\
y &= Ce^{\frac{1}{3}t^3} \\
y &= Ce^{\frac{1}{3}t^3}.
\end{aligned}$$

c. Plug in $y = -1$ into the equation. left side $=\frac{d(-1)}{dt} = 0$, right side$=t^2 - 1 + 1) = 0$. Therefore $y = -1$ satisfies equation is a solution.

d. By linearity principle, general solution of nonhomogeneous equation=general solution of homogeneous equation+a special solution of nonhomogeneous equation=$b + c = Ce^{\frac{1}{3}t^3} - 1$.