Name:

Math 211 **Practice Exam 2**

Directions. *Answer the following questions to the best of your ability. No credit will be given for answers showing no work.*

1. **Consider the following linear systems**

   (a) Calculate the eigenvalues:

   (b) Calculate the eigenvectors:

   (c) Write down the general solution:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ -1 & -5 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

2. **Harmonic Oscillator**
   Consider the equation $\frac{d^2y}{dt^2}+y=0$ for the motion of a simple harmonic oscillator. Let's denote $v=\frac{dy}{dt}$.

   (a) Show that the function $y(t)=-\sin t$ is a solution to this equation.

   (b) What initial condition (at $t=0$) on the yv-plane corresponds to this solution?

   (c) Rewrite the equation as the first-order system with respect to functions $y(t)$ and $v(t)$. Is this system linear? If yes, what is the coefficient matrix of this system?

   (d) Plot the solution curve corresponding to this solution $y(t)=-\sin t$ on the yv-plane. Clearly show the initial conditions point and the direction of the solution curve.

3. **Phase Portraits**
   Given the eigenvalues, λ_i, and corresponding eigenvectors, (x_i, y_i), for the linear system $\frac{d\mathbf{Y}}{dt}=A\mathbf{Y}$. Graph the phase portrait corresponding to each system.

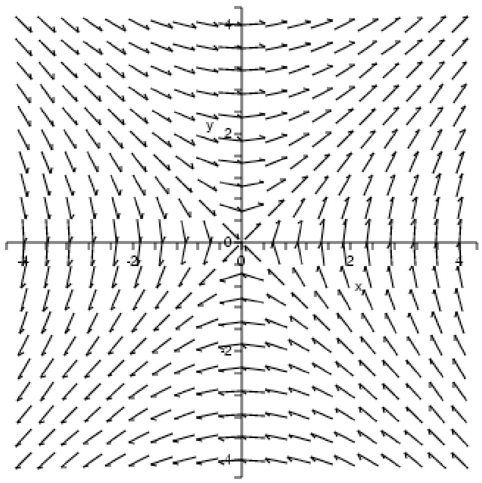
   (a) $\lambda_1=3$, $(x_1,y_1)=(1,-1)$ and $\lambda_2=1$, $(x_2,y_2)=(1,1)$

   (b) $\lambda_1=5$, $(x_1,y_1)=(2,1)$ and $\lambda_2=0$, $(x_2,y_2)=(1,-2)$

   (c) $\lambda_1=2i$, $(x_1,y_1)=(1,i)$ and $\lambda_2=-2i$, $(x_2,y_2)=(1,-i)$

4. **The Phase Plane**

Consider the phase plane of the linear system $\dfrac{d\mathbf{Y}}{dt} = A\mathbf{Y}$



(a) Sketch the solution curve on the phase plane corresponding to the initial condition (2,-3). Then graph the solutions $x(t)$ and $y(t)$ on the $(x, y)t$-plane.

(b) Describe the long-term behavior ($t \to \infty$) of the solution that satisfies the initial condition $(x_0, y_0) = (2, -3)$. Where does $x(t)$ go to when $t \to \infty$? Where does $y(t)$ go to when $t \to \infty$.

(c) What conclusion you can make about the eigenvalues of the coefficient matrix of this system?

(d) What is the equilibrium point(s) for this linear system and classify each as a sink, source, or saddle?

5. **Partially Decoupled Systems**

$$\begin{cases} \dfrac{dx}{dt} & = & xy \\ \\ \dfrac{dy}{dt} & = & y \end{cases}$$

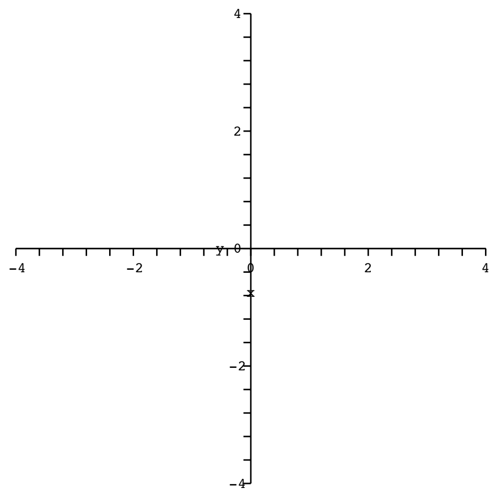
(a) Find the general solution of this system.

(b) Determine the solution of this system which satisfies the initial condition $(x(0), y(0)) = (1, 1)$

6. **Non-Linear Systems**

$$\begin{cases} \dfrac{dx}{dt} &= x + 2y + 1 \\ \\ \dfrac{dy}{dt} &= x - y - 5 \end{cases}$$

(a) Sketch vectors of the vector field of this system at the points $(x, y) = (0, 2), (-2, 0), (-2, 2)$



(b) Find the equilibrium point(s) of this system