SEAS OF SQUARES WITH SIZES FROM A Π_1^0 SET

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Abstract. For each Π_1^0 $S \subseteq \mathbb{N}$, let the S-square shift be the two-dimensional subshift on the alphabet $\{0,1\}$ whose elements consist of squares of 1s of various sizes on a background of 0s, where the side length of each square is in S. Similarly, let the distinct-square shift consist of seas of squares such that no two finite squares have the same size. Extending the self-similar Turing machine tiling construction of [6], we show that if X is an S-square shift or any effectively closed subshift of the distinct square shift, then X is sofic.

The class of multidimensional sofic shifts contains many accessible examples of a geometrical nature. However, since the sofic shifts properly contain the shifts of finite type (SFTs), they also include shifts whose elements encode and make use of arbitrary Turing computations, as pioneered in [17, 2, 13] and applied in [1, 5, 6, 8, 9, 15] and more. For definitions, see Section 1.

Despite this flexibility, the sofic shifts are still a proper subclass of the effectively closed shifts, as evidenced by the mirror shift and examples in [12, 14, 5][1]. It is informative to consider what goes wrong in the following "construction" realizing an arbitrary effectively closed shift A as a sofic shift. Given an element X which may or may not be in A,

(1) Superimpose on X the Turing machine computation which enumerates the forbidden patterns of A.
(2) Whenever a forbidden pattern is enumerated, the computation checks whether this pattern has been used in X.
(3) If the pattern has been used, the computation halts, keeping X out of the sofic shift.

The problem is with Step 2. A Turing computation has only limited access to the bits of X on which it is superimposed. The exact scope of the limitation has not been described, but all examples known to the author of effectively closed shifts which are not sofic were obtained by in some sense allowing elements to pack too much important information into a small area. On the other hand, constructions like the above have been used in [9, 8, 6, 1] in the special circumstance where each element of the shift is known to have constant columns (and in [9, 8], additional geometric properties). In [9], such shifts played a role in characterizing the entropies of the multidimensional SFTs. In [6] and independently [1], it was shown that every effectively closed $\mathbb{Z}^2$ shift consisting of elements with constant columns is sofic. In this paper, we demonstrate the soficness of a class of shifts with a different geometric restriction.

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[1]In [14] the existence of effective $\mathbb{Z}^d$ subshifts with only complex patterns was established; in [5] it was shown that a subshift with only complex patterns cannot be sofic.

Definition 1. *For any $S \subseteq \mathbb{N}$, the two-dimensional S-*square subshift *on the alphabet $\{0,1\}$ is the shift whose elements consist of squares of 1s of various sizes on a background of 0s, where the side length of each square is in S.*

For the sake of being specific let us say that the squares can touch diagonally, but this detail does not matter. Of course, infinite/degenerate squares, such as a whole quarter- or half-plane being filled with 1s, are unavoidably included when squares of arbitrarily large size are. The main theorem is

Theorem 1. *For any Π^0_1 set S, the S-square subshift is sofic.*

The proof expands on the framework developed in [6]. In their construction, arrays of Turing machines operating at multiple scales work together to "read" the common row, compute the forbidden words, and kill any elements whose common row contains a forbidden word. The "reading" process makes strong use of the fact that each bit of the common row is repeated infinitely often along an entire column, so the machines have many opportunities to work together to access it.

In our construction, a similar array of Turing machines "measures" the sizes of the squares in a potential element, computes the forbidden sizes, and kills the element if it sees a square of a forbidden size. However, an extra challenge is that each machine must measure and store the size of every single square in its vicinity, since sizes are not necessarily repeated and thus no machine can compensate for another machine overlooking a square. The higher density of information creates some new difficulties which are summarized at the start of Section 3.

Note that Theorem 1 could not be improved to the statement that every effective sub-subshift of the $\mathbb{N}$-square shift is sofic. For example, take any effectively closed non-sofic shift A on $\{0,1\}$, and embed it into a subshift of the $\{1\}$-square shift by taking each element of A and inserting a row and column of 0s between each original row and column in order to separate them. To avoid this counterexample, we consider subshifts consisting of seas of squares in which no size is repeated.

Definition 2. *The* distinct-square subshift *is the shift whose elements consist of squares of 1s of various sizes on a background of 0s, where the side length of each finite square is distinct.*

With that restriction, similar methods yield the following.

Theorem 2. *Every effectively closed subshift of the distinct-square shift is sofic.*

The S-square shifts can be thought of as a two-dimensional generalization of the one-dimensional S-gap shifts. See [10] for the definition and basic properties, and [7] for an analysis of some computational properties of S-gap shifts for Π^0_1 sets S, including the result that some right-r.e. numbers are not obtainable as the entropy of a Π^0_1 S-gap shift. However, as is often the case when passing from one dimension to multiple dimensions, few of the nice properties of the S-gap shift survive the dimension increase. For example, the entropy of the S-gap shift for all S is well-understood (cf [10]); by contrast, the study of approximations to the entropy of even the $\{1\}$-square shift, better known as the hard square shift, continues to the present day [11]. To the author's knowledge, nothing is known about the the entropies of the S-square shifts more generally. The distinct-square shift has entropy zero.

It is an open question whether the following result of [3] remains true in multiple dimensions: every one-dimensional sofic shift is a factor of an SFT of the same entropy. See [4] for a partial result towards this question and [9] for some discussion. No counterexample comes out of this class of sofic shifts. At the end of the paper we indicate how to modify our construction to produce an SFT with the same entropy as the S-square shift or distinct-square shift it factors onto.

In Section 2 we give a summary of the expanding tileset construction of [6] on which our construction builds. In Section 3 we describe the new algorithm run by tiles at all levels to produce an SFT which factors onto a given S-square shift, proving Theorem 1. The supporting Section 4 provides pseudocode of this algorithm, for reference and also for ease of verifying the claimed runtimes. A short Section 5 describes how to modify the construction of previous two sections to apply to effectively closed subshifts of the distinct square shift, proving Theorem 2. Finally, in Section 6 we describe how to modify both constructions so that they produce an SFT with the same entropy as the subshift they factor onto.

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1. Preliminaries

1.1. **Definitions.** Let G be $\mathbb{Z}^d$ for some positive integer d. In this paper almost always $d = 2$. Let Σ be a finite alphabet. If $D \subseteq G$ is finite, a mapping $\sigma : D \to \Sigma$ is called a *pattern*. If $X \in \Sigma^G$ and σ is a pattern with domain D, we say σ *appears in* X if there is some translation $g \in G$ so that for all $i \in gD$, $X(i) = \sigma(g^{-1}i)$. A subset $A \subseteq \Sigma^G$ is a *subshift* if there is a set of patterns P such that A consists of exactly the $X \in \Sigma^G$ that do not contain any pattern from P. It is a *shift of finite type (SFT)* if there is a finite such P, and an *effectively closed shift* if there is a computably enumerable such P. A subshift $A \subseteq \Sigma^G$ is *sofic* if there is SFT $B \subseteq \Lambda^G$, where Λ is a possibly different alphabet, and a function $\phi : \Lambda \to \Sigma$, such that A is the image of B under ϕ (with notation abused to apply ϕ to an element of Λ^G.) In this situation we say that B *factors onto* A. This was not the original definition of sofic, but it is the most convenient to us. The set P is also called the set of forbidden patterns, and the forbidden patterns themselves are sometimes called local constraints.

Let C be a finite set of colors. A *Wang tileset*, or just *tileset* is any subset $T \subseteq C^4$, whose elements are understood as squares with colored sides, together with the rule that two tiles can be laid adjacent to each other if and only if they have the same color on the edge they share. If it is possible to tile the whole plane with a given tileset, then the set of ways to do this is an SFT contained in $T^{\mathbb{Z}^2}$. The forbidden patterns are simply the 2×1 and 1×2 combinations of tiles with non-matching adjacent edges.

1.2. **Multitape machines simulated with tiles.** Wang [17] introduced a tileset, with one or more of the tiles designated as anchor tiles, such that any tiling of the plane that includes an anchor tile also includes the space-time diagram of a Turing computation. The tiles literally correspond to small fragments of plausible-looking space-time diagram, which when pasted together without discontinuity, inevitably produce a valid computation. The anchor tiles are those displaying the beginning of the tape with the head in the start position and the machine in the start state. The tiles can be made to run any computation, but if the computation halts, an infinite tiling cannot be completed, as there is nothing to go in the next row.

We will need a multitape tileset for what follows, so here we describe how to modify a standard single-tape tileset to get a multitape one. Let there be just one head which reads all the tapes below it. The head retains its ability to move right and left along the bundle of tapes, but it has another possible action: instruct the ith tape to move to the right/left while the other tapes remain fixed. To do this via a tileset/SFT, at the timestep at which the move is triggered, we annotate the entire tape that is to be moved with an L or an R. Local constraints ensure that if one part of a tape thinks it has L or R on it, then all adjacent parts of that tape agree, propagating the signal. Local constraints ensure that the part of the tape right under the head has this annotation exactly when the computation calls for it. Finally, we insist that if a certain tape at a certain timestep is annotated with the instruction to move, then that tape should be shifted in the appropriate direction at the next timestep. Again, this is readily enforced by local checks.

1.3. **Efficient simulation and the runtime-preserving recursion theorem.** In what follows we will design an algorithm which depends for its good runtime on a multitape architecture, so we will declare that our universal Turing machine has multiple tapes. The expression $\varphi_{n,k}(x_1, \dots, x_i)$, where $i \leq k$, denotes the output of this universal machine simulating the nth k-tape program, which receives up to k inputs, one input per tape. We may abbreviate $\varphi_{n,1}(x)$ as $\varphi_n(x)$, in agreement with the standard notation. Let $RT(n, k, x_1, \dots, x_i)$ denote the runtime of the nth k-tape machine on inputs $x_1, \dots x_i$. If the universal machine simulates a machine with fewer tapes than it has, it can always do so with only a constant overhead per step of the simulated machine. To do this, it uses its tapes to mimic the actions of the simulated machine exactly, guided by the additional tape on which the program is written, which is also used to calculate the next simulated move.

By appeal to the Church-Turing thesis, we will often define an algorithm by describing it informally (possibly including implementation details), and then assume we have in hand a machine index n for our algorithm. By appeal to the *smn* theorem[2] we will often informally describe an algorithm which has some parameter e meant to be filled in later, and then assume we have in hand a computable function f such that $f(e)$ is a machine index for the algorithm after parameter e is filled in.

Now we'll observe that a runtime-preserving version of the recursion theorem holds, via the same proof as the regular recursion theorem. The runtime preservation comes at the small cost of one additional tape.

Proposition 1. *For any computable function f and any k, there is an input n and a constant C such that for all inputs $x_1, \dots x_k$, the computations $\varphi_{n,k+1}(x_1, \dots, x_k)$ and $\varphi_{f(n),k}(x_1, \dots, x_k)$ either both converge to the same value or both diverge, and if they both converge then $RT(n, k+1, x_1, \dots, x_k) < C \cdot RT(f(n), k, x_1, \dots, x_k)$.*

Proof. As in the usual proof of the recursion theorem, we let $n = d(v)$ where $d(u)$ is a computable function such that $\varphi_{d(u),k+1}(x_1, \dots, x_{k+1})$ does the following:

(1) If $x_{k+1} \neq \emptyset$, halt.
(2) On the empty $k+1$st tape, calculate $\varphi_u(u)$ (by any method).
(3) Using the $k+1$st tape as the simulation work tape, simulate $\varphi_{\varphi_u(u),k}(x_1, \dots, x_k)$ on the other k tapes with constant overhead, and return its output.

[2]The *smn* theorem that says there is an algorithm for modifying a machine index to replace some of its inputs with hard-coded values; see e.g. [16] for details.

and v is the index such that $\varphi_v(x) = f(d(x))$. Then $RT(d(v), k+1, x_1, \ldots, x_k)$ consists of a constant amount of time for steps (1)-(2), (where step 2 finishes because φ_v is total) followed by $O(RT(f(d(v)), k, x_1, \ldots, x_k))$ steps if $\varphi_{f(d(v)),k}(x_1, \ldots, x_k)$ converges, and divergence otherwise. □

2. Expanding tileset constructions

To establish notation and conventions for our construction, and because we will build on it directly, in this section we summarize the expanding tileset construction of [6]. A tileset T *simulates* a tileset S at zoom level N if there is an injective map $\phi : S \to T^{N^2}$ which takes each tile from S to a valid $N \times N$ array of tiles from T, such that

- For any S-tiling U, $\phi(U)$ is a T-tiling.
- Any T-tiling W can be uniquely divided into an infinite array of $N \times N$ macrotiles from the image of S.
- For any T-tiling W, $\phi^{-1}(W)$ is an S-tiling.

Here we have abused notation to let ϕ map tilings to tilings in the obvious way. Given a sufficiently well-behaved sequence $N_0, N_1, \ldots$, the following construction provides a sequence of tilesets $T_0, T_1, \ldots$ such that for sufficiently large i, each T_i simulates T_{i+1} at zoom level N_i. Therefore, if i_0 is sufficiently large, T_{i_0} simulates each T_{i_0+k} at zoom level $\prod_{j=i_0}^{i_o+k-1} N_j$.

Let s be the number of bits needed to encode all the side colors for a tileset that simulates a universal Turing machine u with four tapes, which we refer to as the program tape, the parameter tape, the color tape, and the extra tape.[3] The side colors of tileset T_i will be binary strings of length $s + 2 + 2\log N_i$. Given a tile, the $4s$ bits obtained by taking the first s bits from each color will be called the "machine part" of the tile; the next 8 bits (2 per color) are the "wire part", and the remaining bits are the "location part". The permitted location parts are as shown in Figure 1(a), where addition is mod N_i. This ensures a unique division of any T_i-tiling into an infinite array of $N_i \times N_i$ macrotiles. The possibilities for the machine and wire parts of a tile depend on the location part of the tile, which corresponds to its position within its $N_i \times N_i$ parent tile. To ensure a layout such as illustrated in Figure 1(b), centrally located child tiles must display legal universal machine tiles with the machine part of their colors; the bottom left tile of this computation region must display an anchor tile.

Any tile that falls within one of the four $(s+2+2\log N_{i+1})$-tile-wide stripes pictured is required to use its wire part as in Figure 1(c) to participate in transmitting a bit between the edge of the macrotile and the bottom edge of the computation region. Any tile at the bottom edge of the computation region acts as the end of the wire, but must verify that the data on its wire bits matches what is written on the "color tape" as displayed by its machine bits. Any bits that are not being used in the ways described should be set to some common neutral value.

One sees that the $N_i \times N_i$ macrotiles will join with each other exactly as tiles with macrocolors of the appropriate length for a T_{i+1} tile would, and the computation region is being set up to do computations on those macrocolors.

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[3]In this first construction the separate tapes are not needed, but we use them so that the main construction can build directly on this one. The "extra tape" is the one which gets swallowed in the runtime-preserving recursion theorem. In [6], polynomial overhead associated to the recursion theorem was acceptable, so they did not need to enter into these details.

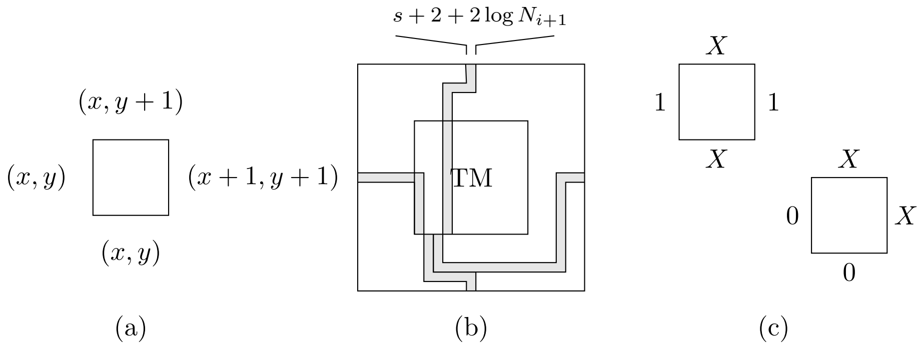


FIGURE 1. (a) Location part of a tile with position (x, y). (b) Overall layout of an $N_i \times N_i$ macrotile. (c) Tiles with wire parts carrying a 1 horizontally and a 0 through a turn.

Let $\varphi_{f(e),2}(i, c)$ be the algorithm which does:

(1) Compute N_i.
(2) Check $|c| = 4(s + 2 + 2 \log N_{i+1})$.
(3) Considering c as the concatenation of four colors, check that it obeys all the restrictions described above.
(4) If the machine part of c gives it a view of any bit of the computation region's parameter tape, check that bit is consistent with "$i + 1$" being on the parameter tape.
(5) As above, check that any visible bit of the program tape is consistent with "e" being the program.
(6) If any checks fail, halt. Otherwise enter an accepting state and run forever.

Note this algorithm runs in poly$(\log N_{i+1})$ time whenever i is the first input, assuming $i \mapsto N_i$ is poly$(\log N_{i+1})$-computable. (We didn't specify a particular scheme for the wire layout, but a poly$(\log N_{i+1})$ scheme certainly exists.) Using the runtime-preserving recursion theorem, let n be such that $\varphi_{n,3}(i, c)$ and $\varphi_{f(n),2}(i, c)$ converge or diverge together, and both reach Step 6 in poly$(\log N_{i+1})$ time. Let $R(i) \in O(\text{poly}(\log N_{i+1}))$ denote the maximum amount of time for the 4-tape universal machine simulation $\varphi_{n,3}(i, c)$, where c is arbitrary. Assuming poly$(\log N_{i+1}) \ll N_i$, let i_0 be large enough that $R(i) < N_i/2$ for all $i \geq i_0$. For $i \geq i_0$, let

$$T_i = \{c : \varphi_{n,3}(i, c) \text{ enters an accepting state.}\}.$$

It is not difficult to prove that each T_i simulates T_{i+1} via the map that sends a T_{i+1}-tile c to the unique macrotile whose macrocolors are given by c. The key is that for $i \geq i_0$, the computation region is large enough to accommodate the entire computation, so a macrotile with given macrocolors c is completed if and only if the computation accepts c. (If the computation halts, there is no way to fill in the computation region, so a macrotile with defective colors c cannot be formed.)

We end the section by recalling the assumptions made on $\{N_i\}_{i<\omega}$, namely that $i \mapsto N_i$ is poly$(\log N_{i+1})$-computable and that poly$(\log N_{i+1}) \ll N_i$.

3. An SFT that factors onto the S-square shift

In this section we give a proof of the main result for the S-square shift. As in [6], we augment the symbols 0 and 1 to make them also into tiles with various possible color combinations. The tiles will be designed to carry out the expanding tileset construction of the previous section, using the extra time in Step 6 to enumerate $\mathbb{N}\setminus S$. The higher the layer of simulated macrotile, the more of $\mathbb{N}\setminus S$ is enumerated, but higher layers are also further removed from the square size information available at the pixel level. To keep the higher level computations aware of the exact square sizes in their vicinity, the "child tiles" making up each "parent tile" make sure that the sizes they know about all get written on their parent's parameter tape. With this information, the algorithm running in a tile can halt its computation, preventing any tiling from being formed, if it finds a forbidden size on its parameter tape. This is the same general strategy as was used in [6] for the case of the effectively closed shift with constant columns.

For readers familiar with [6], the following aspects of this construction have no counterpart in that work. First, each macrotile must keep track of every square in its vicinity, storing roughly $L^{2/3}$ bits of information in each macrotile of pixel side length L. This is much more than the $\log L$ information that needed to be stored in [6]. This large amount of information together with a large required growth rate make it impossible for any one child to have a copy of its parent's complete information, which was the parent-child coordination tool of [6]. We develop a new more distributed communication scheme. Finally, as the input information takes up a significant fraction of the macrotile's tape, polynomial time algorithms no longer suffice; all our algorithms must run in well under quadratic time, necessitating some attention paid to the computation model.

The tiles ultimately produced can be defined as the color combinations accepted by a certain algorithm. Pseudocode for that algorithm is given in Section 4 for reference. Here, we describe the algorithm informally, motivating each of its components as we introduce them.

3.1. **Enforcing a sea of squares.**

The first step is to observe the $\mathbb{N}$-square shift is sofic, a fact which is surely folklore, but we include a proof for the reader's convenience, and because we will build on the SFT Y defined there.

Lemma 1. *The $\mathbb{N}$-square shift is sofic.*

Proof. Consider the alphabet consisting of 0s and the symbols $\leftarrow, \rightarrow \uparrow, \downarrow, ⌞, ⌟, ⌜, ⌝, \circ$. Let Y be the shift whose elements look like seas of squares with nested directed counterclockwise squares drawn inside them, plus limit points (see Figure 2). To see Y is an SFT, we argue we can obtain it by forbidding every 2×2 pattern that does not appear in any element of Y.

These forbidden patterns are enough to enforce that every nonzero symbol (except for $\circ$) must have its line continued on an adjacent symbol with the same orientation, resulting in directed paths with no beginning or end which make only left turns. Therefore, each nonzero component is a union of cycles and/or bi-infinite paths. We claim first that if a nonzero component contains a cycle, it contains one of the patterns ⌜ ⌝ / ⌞ ⌟ or $\circ$. Cycles are partially ordered by spatial inclusion, so pick a cycle which has no cycle inside of it. Since all turns are left turns, every cycle is convex, so a rectangle. By local restrictions, given a corner, only three things

$\uparrow$	$\uparrow$	0	0	0	0	0	0	0	0
$\uparrow$	$\uparrow$	0	$\ulcorner$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\urcorner$	0	0
$\uparrow$	$\uparrow$	0	$\downarrow$	$\ulcorner$	$\leftarrow$	$\urcorner$	$\uparrow$	0	0
$\uparrow$	$\uparrow$	0	$\downarrow$	$\downarrow$	$\circ$	$\uparrow$	$\uparrow$	0	0
$\uparrow$	$\uparrow$	0	$\downarrow$	$\llcorner$	$\rightarrow$	$\lrcorner$	$\uparrow$	0	0
$\uparrow$	$\uparrow$	0	$\llcorner$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\lrcorner$	0	0
$\uparrow$	$\uparrow$	0	0	0	0	0	0	0	0
$\uparrow$	$\uparrow$	0	0	0	0	0	$\ulcorner$	$\leftarrow$	$\leftarrow$
$\uparrow$	$\uparrow$	0	0	0	0	0	$\downarrow$	$\ulcorner$	$\urcorner$
$\uparrow$	$\uparrow$	0	0	$\circ$	0	0	$\downarrow$	$\llcorner$	$\lrcorner$
$\uparrow$	$\uparrow$	0	$\circ$	0	0	0	$\llcorner$	$\rightarrow$	$\rightarrow$
$\uparrow$	$\uparrow$	0	0	0	0	0	0	0	0

FIGURE 2. A permitted pattern from an element of Y

can go diagonally inside of that corner: another corner of the same kind (nested), a $\circ$, or a corner of the diagonally opposite kind. The first case is impossible since it implies another cycle inside this one. The latter two cases correspond to the presence of a $\circ$ and the 2×2 pattern, respectively.

Next, local restrictions guarantee that if a component contains an $n \times n$ square cycle, either that is the outer boundary of the component, or the component also contains a concentric $(n+2) \times (n+2)$ square cycle. Therefore, if a component has any cycle, it is made entirely of square cycles nested in the proper way, and is either a finite or infinite square.

Consider now components with no cycles. Since all turns are left turns, the only possible bi-infinite paths are straight lines, paths with one corner, and paths with two corners. Paths with two corners are impossible by a minimality argument.

If a component contains a path with one corner, there must be another corner of the same kind diagonally nested in that corner, implying an entire path with one corner is nested inside the first path. Therefore, such a component is at least a quarter plane. Local restrictions imply that for each path with one corner, either it is the outer boundary of the component, or there is another single-corner-path nesting with it on the outside. Well-formed quarter-plane or full-plane "squares" result. If component contains a straight line, then by a similar argument it is either a half-plane or a full-plane of parallel lines.

Since each nonzero component is a square, this SFT is Y. By replacing all nonzero symbols with 1s we obtain the $\mathbb{N}$-square shift. □

Going forward it will be convenient to assume that the alphabet of Y includes two versions of each corner symbol, one for use in the interior of squares and one for use on the outer corners. The previous argument is easily adapted to this augmentation. By superimposing tile colors on the symbols of Y instead of the original alphabet, we can assume that we are always dealing with a sea of well-formed squares, and worry only about their sizes.

3.2. **Growth Considerations.** The general strategy outlined so far has placed some restrictions on a the possible rates of growth of the expanding tileset sequence $\{N_i\}_{i<\omega}$ that we will use. After developing an algorithm, just as in the previous section we will select some i_0 large enough and form a tileset T_{i_0} to superimpose (according to some restrictions) on the symbols of Y. Referring to these pixel level tiles as "macrotiles of level i_0", we will arrange that for $i > i_0$, a macrotile at level i occupies a square region whose side length in pixels is $M_i = \prod_{k=i_0}^{i-1} N_k$. Let $L_i = \prod_{k=0}^{i-1} N_k$. Then L_i is an upper bound on the pixel size of the macrotiles at level i, regardless of what we later choose i_0 to be.

Now consider the information which a macrotile at level i must "know" (have written on its parameter tape). We'll say a square is within the responsibility zone of a macrotile if it has at least one full side in the macrotile region. The maximum number of distinctly-sized non-overlapping squares that can appear fully with in a square region of pixel side length L is $O(L^{2/3})$, as one can see by considering the worst situation, in which exactly one square of each size $1, 2, 3, \ldots, m$ appears, where m must satisfy $\sum_{k=1}^{m} k^2 < L^2$. Similarly, the maximum number of distinctly-sized squares that can straddle the boundary of a such a region is $O(L^{1/2})$. Encoding the distinct sizes in a standard binary representation, we will need $O(L_i^{2/3} \log L_i)$ bits available on the parameter tape of each macrotile at level i. Since that macrotile is made of an $N_{i-1} \times N_{i-1}$ array of children, its tape size is $O(N_{i-1})$. So the sequence $\{N_i\}_{i<\omega}$ must satisfy

$$L_i^{2/3} \log L_i \ll N_{i-1},$$

in addition to the $\text{poly}(\log N_i) \ll N_{i-1}$ needed for the expanding tileset part of the construction. The reader can verify that $N_i = 2^{2^{2^i}}$ is fast enough to satisfy this additional constraint, but $N_i = 2^{2^i}$ is not. From here forward we fix $N_i = 2^{2^{2^i}}$. There will be stronger growth requirements for N_i later, but this choice of N_i will work for them.

Note, by the definition of L_i, it cannot be avoided that $N_{i-1}^{2/3} \ll L_i^{2/3} \log L_i$. So whatever algorithm we run to do the consistency checking must run in (non-deterministic) polynomial time, but with the degree of the polynomial strictly less than $3/2$. Some familiar operations which run in linear or $n \log n$ time on reasonable architectures would require quadratic time on a single tape Turing machine, which will not be good enough for our purposes. So we use a multitape machine as described in Section 1.2.

Let us add a couple details to that implementation which are relevant to the current construction. Since there is an edge to our computation region and tapes can be shifted off it, we adopt the convention that any bits that go off the edge are lost and any part of the tape that is just returning from beyond the region is blank. On the left edge, we can assume that algorithms are designed for this and don't move the tapes backward from their original position. On the right edge, it will not cause a problem because if the head writes a symbol, the distance of that symbol from the left edge is less than the time that has elapsed so far (since the head had to get to that location), so the distance of that bit from the right edge is more than the amount of time that is left. A perverse algorithm could possibly lose some of its input if it devoted all its time to shifting the input tape to the right, without even first reading the input. But our algorithm reads its input first.

3.3. **Consistency checking algorithm.** We'll now add more parts to the macrocolor scheme (in addition to the machine, wire, and coordinate parts already being used to create an expanding tileset) in order to satisfy two goals. The first goal is reassuring each individual child that the parent has recorded each size that appeared in that child. The second goal is ensuring that any new large squares, contained in the parent but only fragmentary in each of the children, have their size recorded in the parent. The second goal is easier and we address it first.

3.3.1. *Tracking partial squares.* A child may contain partial squares, such as a corner of a larger square taking a bite from a corner of the child (there can be at most four such partial squares) or the side of a larger square taking off the side of a child (at most two such partial squares).

For any square which has a corner in the child, but does not have a complete side in the child, let us assume the child knows the coordinates (in pixels, relative to the child) and orientation of that corner. Either the partial square is contained in the parent macrotile or not; if it is contained, we want to make sure the parent knows about its size, and if it is not contained we want to make sure the parent knows its coordinates so that the problem is passed along. If a child has a corner, it uses its knowledge of its own position in the parent macrotile to compute the deep coordinates of that corner in the parent macrotile (info of size $O(\log L_{i+1})$). It then uses a new "primary corner message passing part" of the macrocolor to send that information to its two neighbor children who it knows also intersect the same square. Any child that receives such a message must either use it or pass it straight on, stopping only if the edge of the parent tile is reached. If child with a corner receives a message in return, it now has two deep parent corner-coordinates which it can use to calculate the size of this square, which has been confirmed to be contained in the parent. Both children who did this computation now require reassurance from the parent about this size (increasing their number of required reassurances by at most 4 in total; we address the issue of how they get that reassurance below).

On the other hand, any child with a corner that receives no messages back on either arm understands that its corner is also a corner in the parent, and needs to verify that the parent has this corner's coordinates on its parameter tape. To do this we copy a strategy of [6] and introduce a new "deep corner copy part" of the macrocolor. This part of the macrocolor, size $O(\log L_{i+1})$ bits, is intended to display a copy of the part of the parent's parameter tape that contains the deep corner coordinates. (Of course, children on the edge of the parent macrotile should display some neutral default value to any adjacent children of a different parent, displaying the parent information only to their siblings.) To make sure this part of the macrocolor says what it should, each child should check that all (usually) four of their macrocolors agree on this part. And any child whose machine part lets it see a bit of the parent's parameter tape should check that bit is consistent with the deep corner copy part of their macrocolor. Once these checks confirm the accuracy of the information, a child with a corner that is also a corner in the parent can directly check the deep corner copy part of its own macrocolor to confirm that the parent recorded that corner.

Note that an infinite square that covers an entire half-plane is invisible to this message system, and the macrotiles will not know whether that space is filled entirely with 0s or with 1s. Neither case is forbidden, so the macrotile ignorance presents no problem. However, in Section 5, we require the macrotiles to know where they intersect such partial squares, and it is not difficult to augment the message system in order to keep track of them.

3.3.2. *Catching the square at a four-way tile system boundary.* The corner message-passing described above will ensure that a macrotile has on its tape a record of each distinct size of square in its responsibility zone. However, in certain exceptional full tilings there is the possibility that a single square never enters the responsibility zone of any macrotile. This can happen if the macrotiles at every level line up, creating four adjacent pixels which are always at the corners of their respective macrotiles at every scale. A square containing all four of these pixels could never be recorded. To fix this, children located at the corners of each parent macrotile use a new "secondary corner message passing part" of the macrocolor to pass deep corner coordinates to the corner children of adjacent parents. If two such neighbor children have matching corners, they both compute the relevant square's size and both require parental reassurance about that size. If there is a square at a four-way tile system boundary, it will eventually lie completely within four macrotiles, who are corners of their parents, at which point it will be found.

3.3.3. *Reassuring the children.* If we could make a "parent size list copy part" of the macrocolor and enforce its accuracy just as in the last section, this construction would be almost finished, because a child wanting reassurance about a particular size could just look for it in that part of their macrocolor. Here is the problem: the information of the sizes in the parent's responsibility zone consumes on the order of $L_{i+1}^{2/3}$ bits, which is much greater than N_{i-1}, the length of the tapes of the children. So no one child can hold all the knowledge of its parent. Instead, each child will hold some subset of the knowledge of the parent, nondeterministically choosing which of its siblings to share each piece of knowledge with, in such a way that every child receives reassurance that their particular sizes have been recorded by the parent.

Some children, sitting on the parent tape, have direct access to a bit of the parent's information. A group of horizontally adjacent children, knowing that they together hold all the bits describing a single size on the tape of the parent, can use a new "parent size reading part" of the macrocolor to coordinate on the correct value for that size. So we have a situation where each parent size is known by at least one child, each child needs to receive reassurance about up to $L_i^{2/3}$ sizes, and each child can know about less than N_{i-1} sizes.

So we do make a "parent size list" part of the macrocolor, but it carries only a subset of the parent's actual list. We permit the children to display different parent size lists on each of their four sides. Each size on the list comes annotated with a bit to say which direction the information is going. If a child receives parent information at one side, it nondeterministically chooses for each of the three other sides, which of that information is passed on. To ensure that parent information is genuine (not created by children passing a hallucinated size around in a loop, for example), whenever siblings share a size they also annotate it with a counter, which the receiving sibling must increment when passing the size on further. Only the children who are part of a group sitting on the part of the parent tape showing a particular size are permitted to use the "0" counter for that size. Since there are N_i^2 children, an upper bound on the space needed for each counter is $2 \log N_i$ bits.

Since the counter guarantees that received parent information is genuine, if some size is represented in a child but not in the parent, there is no way to reassure that child and a valid tiling is not formed. On the other hand, if there is some way to distribute all of the information subject to the constraints, then the children will nondeterministically find it and produce (perhaps many) valid tilings in which each child is reassured.

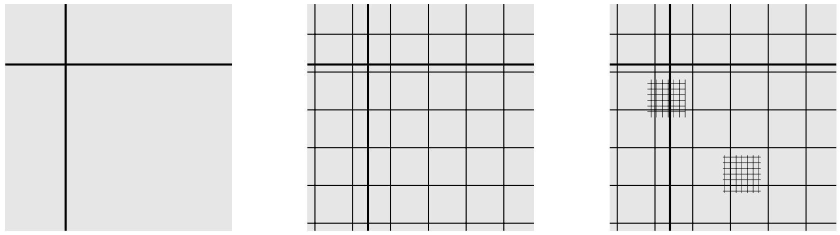
Note that we can ignore the counters when describing a communication scheme; it suffices that for each parent size n and each child that needs to know about n, there is a path of children who know about about n which connects the child to the portion of the parent tape on which n is written. As long as all the children who need to know about n are in one connected component, there is a way to do the counter that is valid, and the children will nondeterministically find it.

3.3.4. *Abstracting the communication problem.* The communication problem described in the previous section can be mostly separated from the details of the tiling construction. Consider a graph with N_i^2 nodes arranged in a square grid, where two nodes are connected by an edge if they are vertically or horizontally adjacent. Now suppose that there are $O(L_{i+1}^{2/3})$ kinds of labels which can be attached to both edges and nodes. Each node can receive at most $O(L_i^{2/3})$ labels. Furthermore, in each $M \times M$ subgrid of nodes, there can be at most $O((ML_i)^{2/3})$ distinct labels appearing on nodes in that subgrid. Here the nodes are the children, the labels on a node are the sizes about which that node needs reassurance, and the $M \times M$ subgrid condition comes from the bound on the number of distinct sizes of square that can appear in an $ML_i \times ML_i$ region. In this graph context, the problem becomes: is there a way to assign labels to edges so that all the nodes with a given label are path connected via edges carrying that label, while limiting the number of labels per edge to $o(\frac{N_{i-1}}{\log N_i})$? This bound on the number of labels per edge takes into account the length of the child's tape $(N_{i-1}/2)$ and the fact that each size consumes $\log L_{i+1} + 2 \log N_i \in O(\log N_i)$ bits of the macrocolor.

In the next section we show that there is always such a labeling. This labeling connects the children who need reassurances about size n to each other, but does not connect them to the parent tape, since the information collected by children on the parent tape does not satisfy the $M \times M$ subgrid condition. But in the labeling we find, each connected component will include an entire row. So the children sitting where n is written on the parent tape can connect to their respective components at a cost of at most one additional piece of knowledge per child by each passing their knowledge directly upward or downward. This completes the description of the algorithm, though we will return to its implementation to gain a runtime improvement in Section 3.5.

3.4. **The plaid.** Returning to the abstracted graph problem of the previous section, we now describe a labeling which solves the problem and uses $O(2^i L_i^{2/3})$ labels per edge. The reader can verify that $2^i L_i^{2/3} \log N_i \ll N_{i-1}$, so this is acceptable.

The scheme can be visualized as a massively multiscale plaid pattern. At the top level, there are up to $O(L_{i+1}^{2/3})$ distinct labels used in the $N_i \times N_i$ grid. Apply labels to all the vertical edges in a pattern of dense stripes: each edge in the first column receives the first $L_i^{2/3}$ distinct labels, each edge in the second column receives the next $L_i^{2/3}$ distinct labels, and so on. When the distinct labels run out (as they will since $L_{i+1}^{2/3}/L_i^{2/3} = N_i^{2/3} \ll N_i$) return to the beginning of the list of distinct labels and repeat. Do the same with horizontal edges. So far each edge has $L_i^{2/3}$ labels. This was the first layer of plaid.



(a) (b) (c)

FIGURE 3. Edges carrying a fixed label in a square subgrid of side length $\sim N_i^{2/3}$. (a) Labels from the top layer of plaid only (layer 0). (b) Labels from layers 0-1. (c) Labels from layers 0-2; we see that of the many $\sim N_i^{4/9} \times N_i^{4/9}$ subsubgrids, only two of them have nodes which bear the given label.

Next partition the grid into square subgrids whose side length is long enough that for every label, there is a vertical and horizontal line of edges with that label in each subgrid. Such a division creates subgrids with length on the order of $N_i^{2/3}$. By the subgrid condition, the number of distinct labels which have been applied to nodes in that subgrid is $O((N_i^{2/3} L_i)^{2/3})$, or $O(N_i^{4/9} L_i^{2/3})$. Now we repeat the process inside each subgrid, adding $L_i^{2/3}$ more labels per edge within the subgrid, this time using only labels which have been applied to nodes in that subgrid. Again this results in a plaid pattern within the subgrid with each label appearing on a horizontal and vertical stripe of edges in every subsubgrid with length on the order of $N_i^{4/9}$. All the edges with a given label remain connected since the new smaller scale plaid intersects the larger scale plaid; see Figure 3.

Continuing in this way, we can bound the number of steps it takes to get down to some fixed small subgrid size (at some point the implicit rounding in our calculations starts to matter and we take that point). The subgrid size at step j is $O(N_i^{(2/3)^j}) = O(2^{2^{2^i}(2/3)^j})$, so the number of steps is bounded by $O(2^i)$. Once we reach a fixed small size, we stop with the plaid and just put all the labels from the nodes in that base grid onto all of the edges of that grid, which adds an additional finite multiple of $L_i^{2/3}$ labels to each edge. In this way, all the nodes with a given label are connected to all other nodes with that label with a total of $O(2^i L_i^{2/3})$ labels per edge.

3.5. **Runtime considerations.** Each macrotile at level i has a parameter tape with $O(L_i^{2/3} \log L_i)$ bits of information on it, the bulk of which is a long list of sizes of squares contained in that tile. The color tape of that tile is also filled with macrocolor information, most of which is $O(2^i L_i^{2/3} \log N_i)$ bits worth of parental sizes, annotated with counters and some indication of the direction of information flow.

The algorithm the tile runs to determine if the macrocolors behave appropriately needs to finish in $\ll N_{i-1}$ steps in order for each level to simulate the next properly and still have time left over for enumerating $\mathbb{N} \setminus S$. The bulk of the time is spent dealing with those very long lists of sizes, so in this section we show how to make that run close to linear in the number of sizes. (Recall that it is necessary to keep the algorithm strictly faster than polynomial exponent 3/2 in the number of sizes.) The time taken by various other steps (expanding tileset, deducing new sizes from corners) is small in comparison; these little runtime contributions are collected explicitly in the pseudocode of the next section for reference.

For each size appearing on the parameter tape, it must also appear in some macrocolor on the color tape; for each size that appears on the color tape, all (up to 4) of its macrocolor appearances must be accompanied by counters that increment appropriately. If all (up to 5) mentions of a particular size could be physically colocated, then checking that single size for compliance would take $\mathrm{poly}(\log N_i)$ time, so the total time spent checking would be $2^i L_i^{2/3}\mathrm{poly}(\log N_i)$, which is in fact the runtime we're aiming for. But if the mentions are not colocated, even checking a single size could take $2^i L_i^{2/3} \log N_i$ time, most of which is just for the head to travel between mentions.

As a first step to colocation, we modify the universal machine to have four color tapes, one for each macrocolor, so that the macrocolors are essentially superimposed instead of concatenated. We also give the universal machine an additional tape, bringing its total number of tapes up to 8, so that our algorithm can be a 6-tape algorithm that has a work tape and treats the color and parameter tapes as read-only. Next, we declare (and use the algorithm to check) that the lists of sizes on the parameter tape and in the macrocolors must be sorted and have no repeat sizes. We do not have to do the sorting ourselves since the nondeterminism can take care of it. Still the original problem remains: the lists contain different sizes and may be badly misaligned. This can be fixed by allowing the tapes to move independently. The point is that now a single instruction can move $O(2^i L_i^{2/3} \log N_i)$ many bits at once, with corresponding runtime savings.

The algorithm looks at the beginning of five superimposed lists (4 macrocolors and one parameter tape), which have been moved so that their first entries are aligned, and checks on the least size n it sees: if it is on the parameter tape does it appear on a color tape? And do the counters on the color tapes behave appropriately? During this time it is ignoring any lists whose first size is larger than n. If all that checks out, the head moves forward one data unit, bringing forward with it all the tapes which it was ignoring on the last step. This whole combo-step takes time $\mathrm{poly}(\log N_i)$, and makes the algorithm ready to check the next smallest size. After $O(2^i L_i^{2/3})$ repetitions of this, all the necessary checks have been made. Thus with a multitape machine, the total runtime for this step is $2^i L_i^{2/3}(\mathrm{poly}(\log N_i))$, as desired.

4. The algorithm for the SFT

The algorithm run by macrotiles at all levels is defined as follows. Let $\varphi_{f(e),6}(p, c_1, c_2, c_3, c_4)$ be the algorithm which does the following (considering all inputs as strings).

Initialize:

(1) Start reading p, expecting first two numbers: i_0, i with $i_0 \leq i$.
(2) Calculate $N_i = 2^{2^{2^i}}$, $L_i = \prod_{j=0}^{i} N_j$.

Data Validation:

(3) Check that the rest of the data on p is:

   (a) Up to four deep coordinates, with corner orientation information, relative to this tile. Length limit: $8\log(L_i)$ bits
   (b) A list of sizes. Length limit: $O(L_i^{2/3}\log L_i)$ bits. (The appropriate constant from the O-notation is hard-coded into the algorithm.)

   If the length limit is exceeded, halt. Time: $O(L_i^{2/3}\log L_i)$

(4) Check that the data on each c_k is:
   (a) Machine parts and wire parts. Length: $O(1)$.
   (b) Coordinates of this tile in the parent. Length: $2\log(N_i)$.
   (c) Deep corner copy part: up to four deep coordinates, relative to the parent. Length limit: $8\log L_{i+1}$.
   (d) Primary corner message passing part. Up to four deep coordinates, relative to the parent, two incoming and two outgoing. Length limit: $8\log L_{i+1}$
   (e) Secondary corner message passing part. Up to two deep coordinates, relative to this tile, one incoming and one outgoing. Length limit: $4\log L_i$
   (f) Parent size reading part. Length: $\log L_{i+1}$.
   (g) Parent size list part. A list of sizes, each less than L_{i+1}, each size annotated with a counter less than N_i^2, and a flag that says whether that size is incoming or outgoing. Length limit: $O(2^i L_i^{2/3}\log L_{i+1})$.

   If the length limit of any part is exceeded, halt. Time: $O(2^i L_i^{2/3}\log L_{i+1})$.

Dealing with corners:

(5) Check the machine, wire and coordinate parts just as in the expanding tileset construction. If the location of this tile lets us see a bit of the parent tapes at time 0, check that i_0 and $i+1$ are consistently on the parameter tape and e is consistently on the program tape. Time: poly($\log N_i$), using the fact that the relevant areas on the four macrocolors are colocated.
(6) Check that all four deep corner copy parts match (or are neutral valued, as appropriate for some sides of a tile on the edge of its parent). If the location of this tile lets us see the deep corner part of the parent parameter tape, check consistency with the deep corner copy part. Time: poly($\log L_{i+1}$).
(7) If there were any corners on the parameter tape, use the location of this tile as well as $M_i = \prod_{j=i_0}^{i-1} N_i$, the pixel length of this tile, to calculate the corner's deep coordinates relative to the parent. Check that the appropriate two macrocolors are displaying these coordinates as outgoing messages in the primary corner messaging part (unless doing so would display a message outside the parent tile). If there are matching incoming coordinates, use them to calculate the size of the square, and check that this size appears on one of the parent size lists. Time: poly($\log N_i$) for calculations, plus $O(2^i L_i^{2/3}\log N_i)$ for looking for the matching length.
(8) If this tile is located in a corner location in the parent and also has a corner (two different senses of the word "corner"), and if the orientation is such that both arms of the corner exit the parent, repeat the previous step with appropriate variation to communicate with adjacent non-sibling children through the secondary corner messaging part. Time: $O(2^i L_i^{2/3}\log N_i)$.
(9) If a corner in this tile has all its messages unanswered, make sure that corner appears on the deep corner copy part. Time: poly($\log L_{i+1}$).

(10) If there is an incoming message in the primary corner message passing part of one macrocolor, and it does not match any corner this tile has, check the same message is outgoing in the same channel on the opposite side (unless doing so would cause the message to display on the outside of the parent tile). If there is an incoming message in the secondary corner message passing part with no matching corner in this tile, it is ignored. Time: $O(\log L_{i+1})$.

Dealing with reassurances:

(11) If the location of this tile lets us see a bit of the size list of the parent parameter tape, check that the parent size reading parts are displaying the same thing on the left and the right (or possibly just displaying on one side, if this tile is located at one edge of a parent size on the tape). Check that the parent size being displayed is consistent with what is written on the tape. If the location of this tile does not let us see the parent tape, the parent size reading parts should be empty. Time: $\text{poly}(\log L_{i+1})$.

(12) Check that the size lists on both p and each macrocolor c_k are sorted. Runtime: $O(2^i L_i^{2/3} \log(L_{i+1}))$.

(13) If any size from the parent size list of any macrocolor c_k is shown as outgoing with counter 0, check that this size also appears in the parent size reading part. Runtime: $O(2^i L_i^{2/3}(\log L_{i+1}))$, most of which is travel, because there can be at most four uses of counter 0.

(14) For all sizes listed in any macrocolor c_k, or on the tape p, do the following:
   (a) If the size is listed in a macrocolor, verify that there is exactly one macrocolor on which that size is incoming, and that the counter annotating the size on any outgoing sides is one more than the incoming counter. Exception: it is ok to have no incoming size if all outgoing sizes are annotated with 0.
   (b) If the size is listed on the parameter tape p, make sure it is listed in some macrocolor.

   Time: $2^i L_i^{2/3}(\text{poly}(\log L_{i+1}))$, assuming a multitape machine.

Killing tiles with forbidden sizes:

(15) Having successfully completed all these checks (halting to kill the tiling if they fail), run the algorithm which enumerates the complement of S. Whenever a size is enumerated there, check to see if it appears on p's list. If it does, halt to kill the tiling.

By the runtime-preserving recursion theorem, let n be a fixed point such that $RT(f(n), 6, p, c_1, \ldots, c_4)$, $RT(n, 7, p, c_1, \ldots, c_4)$, and the universal simulation of the same coincide to within a constant factor. Let $RT(i)$ denote the maximum time it could take for the simulation to reach Step 15 given that p starts with i_0, i for any $i_0 \leq i$. We see that $RT(i)$ is $2^i L_i^{2/3} \text{poly}(\log N_i)$. Fix i_0 large enough that $RT(i) \ll N_{i-1}$ for all $i \geq i_0$.

Now define an SFT B as follows. For any symbol s from the alphabet of Y, define p_s as follows. If s is not an outer corner, p_s is i_0, i_0. If s is an outer corner, p_s is $i_0, i_0, \langle (0,0), r \rangle$, where r indicates the orientation of s. The alphabet of B consists of those combinations s, c_1, c_2, c_3, c_4, where s is a symbol of Y, such that $\varphi_{n,7}(p_s, c_1, c_2, c_3, c_4)$ makes it to step 15 without entering the kill state. The forbidden patterns of B, of course, are the forbidden patterns of Y, together with the tiling restrictions declaring that adjacent colors must match.

This completes the pseudo-code for the construction. The main points of the verification are convincing oneself that (1) each permitted pattern of the S-square shift has a valid B-preimage, (2) whenever a Y-pattern has a valid B-preimage, each macrotile of the B-preimage has on its parameter tape a list of all sizes of square within its responsibility zone (including sufficiently small squares intersecting at a corner, if they are also contained in the Y-pattern), and (3) for each n, there is a macrotile size large enough that no macrotile that large with a forbidden size less than n on its parameter tape can be formed. The details of the verification are numerous but not difficult.

5. Effectively closed subshifts of the distinct-square shift

The previous argument needs only slight modifications in order to prove Theorem 2, that any effectively closed subshift of the distinct-square shift is sofic. We slightly increase the information monitored by each macrotile, to ensure that each macrotile contains a complete description of what is going on in its responsibility zone as defined below. Then as it uses its extra time to enumerate an arbitrary set of additional forbidden patterns, it can check to see whether each pattern appears in its responsibility zone, and kill the tiling if so. In Section 5.2, we describe how to do this check efficiently.

5.1. **Adjusting each macrotile to keep track of slightly more information.** First, require that the parameter tape of each macrotile, in addition to storing the list of distinct sizes appearing within its responsibility zone, must also annotate each size with the deep coordinates of the lower left hand corner of the unique square with that size. (These deep coordinates may lie outside the macrotile for some squares, which is fine.) These annotations are passed among the children, and any child needing reassurance about a particular size also needs reassurance about the exact location of the square with that size. Since the tape is only permitted to hold one set of deep coordinates per distinct size, this prevents repeated sizes from occurring within the responsibility zone of a given macrotile. To prevent the parent from hallucinating additional sizes, we additionally require the parent to receive confirmation that each of its recorded sizes is actually realized in a specific child, for example by dictating that when a child receives distributed parental information, it must either use it or pass it on. Lastly, require each macrotile to also keep track of the location of any partial sides within its responsibility zone. By a partial side, we mean a row or column of 1s whose corners are not within the responsibility zone, such as what appears on the left edge in Figure 2. The amount of information now needed on the tape at level i is essentially unchanged, still $O(L_i^{2/3} \log L_i)$.

Second, the responsibility zones must be expanded so that any finite region eventually lies within the responsibility zone of a single macrotile. Require each child tile that is on the boundary of its parent macrotile to share its parameter tape with its neighbor(s) of a different parent via a new part of the macrocolor, and to require reassurance from its own parent about any information it receives in return. Children at the corners of four distinct parent tiles cooperate to share all their information with each other and require additional reassurance about all of it. In this way, the responsibility zone of a given macrotile extends beyond its area by at least the width of one of its child tiles in every direction, but not more than double the width of one of its child tiles. The result: even in an exceptional tiling, every finite region is eventually within the responsibility zone of a single macrotile. The amount of information needed on the parameter tapes and macrocolors is increased by at most a constant factor. Note that a macrotile is now tracking deep partial corner coordinates for corners located in its entire responsibility zone, including possibly corners that are outside that macrotile, and same holds for coordinates of partial sides. Again, the method of encoding the deep coordinates can be extended to accommodate this. The macrotile's tape now contains a complete description of what is in its responsibility zone.

5.2. **Recognizing when a given forbidden pattern appears.** Without loss of generality, we may assume that the enumerated forbidden patterns all have square domain. We may also assume that each forbidden pattern has at least one 0 and at least one 1, since if a solid square of 0s or 1s is forbidden, there is one with minimal size, and it can be forbidden manually. Since the tileset already forbids patterns that do not appear in the $\mathbb{N}$-square shift, we may assume all forbidden patterns consist of squares and partial squares of 1s. For each forbidden pattern, there are just countably many ways to complete its partial squares, so we can assume each forbidden pattern is enumerated infinitely many times, each time annotated with a particular choice of size and location (relative to the domain of the forbidden pattern) of each of its partial squares. To account for squares of infinite size, "Large" is a permitted size, and "Far" is permitted coordinate information for "Large" squares. Observe that if a pattern is forbidden, it is correct to forbid all possible ways of completing it, and conversely, if all ways of completing it are forbidden, the pattern itself cannot appear.

It is efficient to recognize whether an annotated pattern appears in the responsibility zone. Assuming first that the annotated pattern contains at least one finite square or partial corner, observe that if that component is located in the parameter list, its location in the macrotile uniquely specifies a single candidate for the location of the forbidden pattern relative the macrotile. So make one pass through the parameter tape to look for such a component. If the location-fixing component is not found, or if its position implies the location of the forbidden pattern extends outside the responsibility zone, or if the annotated pattern references finite sizes that are larger than what the macrotile tracks, the annotated pattern is safely ignored (a larger macrotile will consider it). Otherwise, make a second pass through the parameter list, this time asking each component whether it intersects the candidate region, and if so, whether the annotated forbidden pattern thinks it should be there. At the end of the second pass, if no extraneous components have intruded on the candidate region, and if all components of the annotated pattern have been checked off, then the forbidden pattern appears and the macrotile is forbidden. For a macrotile at level i, these checks take time $rL_i^{2/3}\operatorname{poly}(\log L_i)$, where r is the size of the annotated pattern.

Suppose now that the annotated pattern contains no square of finite size, nor a partial corner. The only patterns that can do this have one or two "Large" partial sides, with all their coordinates "Far", and no other squares. We assume such patterns are split into further annotated versions, each specifying the relative location of some additional finite square or partial corner, plus one annotation asserting that any such square or corner is "Far". The former case is then handled as described above, and in the latter case the macrotile considers the annotated pattern found if it has matching partial side(s) and no finite sizes or partial corners on its tape.

The second pass ensures that in every case in which a macrotile is killed, it was appropriate to do so. Conversely, if a configuration contains a given forbidden pattern, it contains it in a context which is covered by one of the annotations, so that configuration will be killed if that annotation is ever considered by a macrotile which contains both the forbidden pattern and enough of the context. And each annotation will be eventually considered, because the time to get through the kth annotation in a macrotile at level i is $c_k + (\sum_{j\leq k} r_j)L_i^{2/3}\operatorname{poly}(\log L_i)$, where c_k is the computing time to enumerate the first k annotations, and r_j is the size of the jth annotation.

This completes the sketch of modifications needed to prove Theorem 2.

6. An SFT with the same entropy as its factor

For any $\mathbb{Z}^2$ subshift A, let $A\restriction n$ denote the set of patterns on an $n\times n$ rectangular domain $D\subseteq\mathbb{Z}^2$ which appear in some element of A. The *entropy* of A is defined as

$$h(A)=\lim_{n\to\infty}\frac{\log|A\restriction n|}{n^2}.$$

The SFTs described in the previous sections had more entropy than the shifts they factored onto because in each macrotile, the children had many choices about how to route the information of parental sizes. Also, nothing prevented a macrotile from believing it had sizes and/or partial corners which it did not have. We show how to modify the construction to force the tiles to claim only the sizes they must, and force the internal communication of the children to follow a specific and unique plaid protocol. In this modification, there are no counters; hallucinatory parental reassurances are made impossible by the rigid nature of the protocol.

Let A be an S-square shift or an effectively closed subshift of the distinct square shift. Let B' denote the modified SFT which we will describe below, with Λ' its alphabet. Let $F_n\subseteq\mathbb{Z}^2$ be any $n\times n$ square. Let ∂F_n be its boundary. Let M_i denote the side lengths of the macrotiles at level i. We will arrange the following unique extension property: for each M_i, and each pattern $\sigma: F_{M_i}\to\{0,1\}$ that appears in an element of A, and each pattern $\tau:\partial F_{M_i}\to\Lambda'$ that is consistent with σ, there is at most one way to extend τ to all of F_{M_i} so that the result is a single level i macrotile consistent with σ. This is enough to ensure that $h(B')=h(A)$, because

$$|B'\restriction M_i|\leq M_{i-1}^2|\Lambda'|^{4M_iM_{i-1}+4N_{i-1}^2M_{i-1}}|A\restriction M_i|,$$

where the factor of M_{i-1}^2 accounts for all the possibilities of how the $(i-1)$th layer of macrotiles can be aligned, and the $|\Lambda'|^{4M_iM_{i-1}+4N_{i-1}^2M_{i-1}}$ factor accounts for all the possible ways to fill in symbols of Λ' completely on partial layer-$(i-1)$ tiles at the edges of the region and on the boundary of layer-$(i-1)$ tiles fully contained within the region. Taking logs, dividing by M_i^2, and taking the limit as $i\to\infty$ completes the entropy calculation. Now we show how to satisfy the unique extension property.

The partial corner lists are easy to make unique: we can restrict the partial corner part of the parameter tape to appear in lexicographic order; also, when a child sees its deep corner copy part include a corner at that child's location, that child must verify that it truly has an unrequited corner there.

To pin down the protocol of parent size list communication among children, fix in advance a division of the parent macrotile into $O(2^i)$ layers of subgrids large enough to accommodate the worst case of the plaid described in Section 3.4. In this way each child tile can know its location in each subgrid to which it belongs. The macrocolors, instead of containing one long list of parent sizes, will now contain $O(2^i)$ sorted lists, one associated to each layer of plaid. The child tiles will perform checks on these lists in order to force a plaid protocol similar to that of Section 3.4, with a few modifications to make it possible to enforce that a size is used in a particular subgrid if and only if some child in that subgrid needs it.

6.1. **Forcing a plaid pattern at each layer.** To enforce a local stripes pattern, children verify that within each subgrid to which they belong, the sizes associated to that region are propagated straight horizontally and vertically within the subgrid, and not propagated outside of it. To ensure that the same stripes are repeated horizontally and vertically within a given subgrid at layer j, the children who lie on the main diagonals of its subgrids at layer $j+1$ can check that their horizontal and vertical jth lists are in exact agreement. To enforce the order of sizes in the plaid, the children just below all the main diagonals of the layer $j+1$ subgrids verify that their layer j horizontal sizes must all be larger than their layer j vertical sizes. Since the number of sizes used in the given layer j subgrid may be much less than than $L_i^{2/3}$ times the width of the $(j+1)$st layer, we allow a filler symbol in the size lists. By insisting that all lists be full, and that layer j sizes never follow filler symbols within layer $j+1$ subgrids, we obtain well-formed plaids in each subgrid, with each size occupying exactly one row and one column in each subsubgrid. The different possibilities now correspond only to different choices of what sizes should go in the plaid in each subgrid.

6.2. **Making exactly the right sizes show up in each subgrid.** We use the ordering of the plaid to ensure that the sizes that are listed in any subgrid of layer $j+1$ are a subset of those in its parent subgrid in layer j. To check this, the children compare their $(j+1)$st list to their jth list, and if some n on the $(j+1)$st list lies between the least and greatest elements of the jth list (or between 0 and its greatest element, for children in the upper left corner of a layer $j+1$ subgrid), then n needs to be on the jth list or the child kills the tiling, since n cannot legally appear in the parent subgrid anywhere else. This leaves an edge case where a size bigger than the largest size of one layer j list and smaller than the smallest size of the next could hallucinate itself into the $(j+1)$st layer. To fix this, we modify the plaid convention to insist that the size lists in neighboring columns and rows within each subgrid are not disjoint, but overlap by exactly one entry, which is the least element of one list and the greatest element of the other. This ensures that no new sizes are added when passing from layer to layer.

To ensure a size appears in a subgrid only if it needs to be there, each size that appears in a given subgrid must be annotated with the coordinates of the lexicographically least child in that subgrid who needs that size. So for example, in the topmost layer of plaid, the relevant region is the entire parent macrotile, and so each size represented in the top layer of plaid is annotated with the least child in the entire macrotile who needs that size.

First the children need to enforce that if a particular size n appears in one layer of plaid with annotation a, this size-annotation pair must be propagated into the subgrid in which a is located in the next layer. To do this, for each $\langle n, a\rangle$ on a given child's jth list, if that child is located in the same $(j+1)$st-layer subgrid as a, and if n is between the least and greatest elements of the child's $(j+1)$st list, then $\langle n, a\rangle$ must appear on the $(j+1)$st list. If that child is not located in the same $(j+1)$st layer subgrid as a, but n still appears on its $(j+1)$st list, it must appear as $\langle n, b\rangle$ for some b lexicographically greater than a which is in the same $(j+1)$st subgrid as this child. (If everything in the child's $(j+1)$st subgrid is lexicographically smaller than a, this is a ban on n propagating here.)

Finally, we need to check that all claimed annotations are accurate. If the child with location a has $\langle n, a\rangle$ on its last list, it must verify that it actually needs n or kill the tiling. And each child lexicographically less than a which sees $\langle n, a\rangle$ on its last list must verify that it does not need n, or kill the tiling.

As before, there is a algorithm for doing all these checks that is linear in the number of sizes.

6.3. **Connecting the top layer to the parent tape.** The restrictions ensuring a unique plaid have also ensured that the top layer of plaid contains exactly the sizes needed by the children. To make sure the contents of the parent tape correspond exactly to the contents of the top layer of plaid, a new part of the macrocolor is employed, which holds a size-annotation pair. The first child of each parent size reading group must use this channel to pass the size it has read straight down, along with an annotation indicating exactly at what location the message will find that size in the top layer of plaid. Any child receiving such a message must pass it straight onward, or verify it, if that child is the one named in the annotation. This ensures everything on the parent tape makes it into the plaid. To ensure everything on the plaid came from the parent tape, declare that every size in the top layer of plaid must carry an additional annotation, namely the same location where the message from the parent tape will meet the plaid; the child at that location can check that the message from the tape really arrives.

6.4. **Unique extension property.** To see that the unique extension property holds, one can argue inductively that the following stronger property holds at all levels: for any valid A-pattern σ on F_{M_i} and any consistent B'-pattern τ on its boundary, not only is there at most one consistent extension of τ to all of F_{M_i} that produces a single level i macrotile, but also that when such a macrotile exists, its parameter tape must contain exactly the sizes that appear in the macrotile's responsibility zone, as well as exactly the deep coordinates of corners of partial squares which extend outside the responsibility zone. Since the responsibility zone of the macrotile extends outside the A-pattern, this may at first glance appear not well-defined. However, if τ can be extended to form a single macrotile on σ, then the information τ displays outside the macrotile can be interpreted as message-passing which uniquely determines what is going on in the responsibility zone, in the sense that there is only one extension of σ to the entire responsibility zone which allows τ to also be extended there.

Assuming the stronger property holds at level i, its proof sketch for level $i+1$ goes as follows. Given σ, divide it into N_{i-1}^2 child tiles of side length M_{i-1}. For each child tile, its coordinates in the parent are determined. Using bits of σ for interior edges and messages from τ for exterior edges, determine what is going on in each child tile's responsibility zone. Assuming each child tile will be completed, its parameter tape is thus determined. The corner messages it sends out are uniquely determined by its parameter tape, and those it receives are uniquely determined by its adjacent sibling's parameter tape or by τ. The level i tile's deep corner coordinates are therefore uniquely determined as the only ones that all the child tiles will accept, namely the "correct" ones. Now that the corner calculations have been established, it is also pinned down which children will ask for reassurances about what sizes. The unique plaid protocol uniquely determines what will go in the parent size list part of each child's macrocolor in order to provide those reassurances, as well as guarantees that the parent tape size list part contains exactly the sizes requested by the children. The parent parameter tape is now completely fixed, when you count the fact that the algorithm running on it will kill any tiling in which it is not appropriately ordered. This also fixes the parent size reading parts of each child tile. Since the inputs to the color tapes of the parent come from τ and the computation is deterministic, the wire and machine parts of all the children are also uniquely determined. So every child's macrocolors are uniquely determined. This completely determines what symbols of Λ' can go on the boundary of each child tile, because the only messages that leave the child tile (at any level of its grandchildren) are secondary corner passing messages or responsibility zone information sharing messages, the content of which is uniquely determined from σ and τ. By induction, the rest of the symbols of Λ' are uniquely determined. If any stage of this process was impossible to carry out, then there is no extension of τ to produce all of σ. So the number of possible τ-extensions is either 0 or 1.

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